Sigma Journal of Mathematical and Statistical Sciences (SJMSS)

Published by Altech Press • https://sjmss.altechpress.com

Vol. 1, No. 1 (2026), pp. 43–62

Research Article

Intelligent Parameter Estimation for NHPP Models with Inverse Weibull Intensity Using a Hybrid ANN–ABC Approach

Adel S. Hussain [1], Ala Amourah [2,3], Abdullah Alsoboh [4], Hasanain Jalil Neamah Alsaedi [5], Mohammad A. Tashtoush [6,7*], Rana A. Almuttalibi [8]

[1]*Department of IT, Amedi Technical Institutes, University of Duhok Polytechnic, Duhok, Iraq.*

[2]*Mathematics Education Program, Faculty of Education and Arts, Sohar University, Sohar 311, Oman; AAmourah@su.edu.om.*

[3]*Jadara University Research Center, Jadara University, Jordan.*

[4]*College of Applied and Health Sciences, A'Sharqiyah University, Post Box No. 42, Post Code No. 400, Ibra, Sultanate of Oman; abdullah.alsoboh@asu.edu.om*

[5]*Business Information Technology Department, College of Business Information, University Information Technology and Communication, Baghdad, Iraq.*

[6]*Department of Basic Sciences, AL-Huson University College, AL-Balqa Applied University, Salt, Jordan.*

[7]*Faculty of Education and Arts, Sohar University, Sohar, Oman.* tashtoushzz@su.edu.om

[8]*Office of University President, University of Information Technology and Communications, Baghdad, Iraq. rana.aziz@uoitc.edu.iq*

*** Corresponding author:** tashtoushzz@su.edu.om,

Received: 15 August 2026 • Revised: 21 September 2026 • Accepted: 22 September 2026 • Published online: 30 September 2026

ABSTRACT

This study proposes an intelligent hybrid parameter-estimation framework for modeling the time-dependent failure behavior of repairable systems using a Non-Homogeneous Poisson Process (NHPP) with an Inverse Weibull Process (IWP) intensity structure. Accurate estimation of NHPP parameters is essential for reliability analysis, predictive maintenance, and event forecasting; however, conventional maximum likelihood estimation (MLE) often encounters difficulties when the likelihood surface is highly nonlinear or multimodal. To address this limitation, a hybrid Artificial Neural Network–Artificial Bee Colony (ANN–ABC) estimator is developed, where the neural network captures complex nonlinear relationships while the Artificial Bee Colony algorithm performs global optimization of the network parameters to improve estimation accuracy and convergence stability. The proposed framework is evaluated through an extensive Monte Carlo simulation study using multiple sample sizes and is further validated using a real clinical dataset comprising 299 heart-failure patients. The performance of the proposed estimator is compared with classical MLE and Particle Swarm Optimization (PSO) using Root Mean Squared Error (RMSE) and Bayesian Information Criterion (BIC) as the primary evaluation metrics. The numerical results consistently demonstrate that the ANN–ABC estimator provides lower RMSE and BIC values than the competing methods, with the greatest improvements observed for medium and large sample sizes. These findings indicate that the proposed

hybrid estimation strategy offers a robust, accurate, and computationally efficient alternative for NHPP parameter estimation. The proposed framework has promising applications in predictive maintenance, reliability engineering, healthcare event modeling, and intelligent decision-support systems. Future research will investigate Bayesian inference and imperfect repair models to further enhance uncertainty quantification and predictive performance.

Keywords: Inverse Weibull Process; Non-Homogeneous Poisson Process (NHPP); Artificial Neural Network (ANN); Artificial Bee Colony (ABC) Algorithm; Reliability Modeling; Predictive Maintenance; Failure Data Analytics.
MSC 2020: 62N05; 60G55; 68T07; 90C59

1. Introduction

Predictive maintenance optimization and close analysis of failure in repairable systems and smart monitoring, and particularly in safety-bound fields like healthcare, power systems, and infrastructure involving the IoT-based system require reliability models drastically. The Non-Homogeneous Poisson Process (NHPP) is a widely used framework for modeling time-varying failure rates in repairable systems, including reliability-growth and power-law process formulations. In particular, extended reliability-growth models have been developed to accommodate corrective actions during system development [1], while maximum-likelihood and numerical procedures have been investigated for estimating the parameters of power-law NHPP models from exact failure-time data [2]. In this regard, the Inverse Weibull Process (IWP) gives an adaptable solution of modeling-tired processes in systems where hazards deceased at increasing speeds and loss of life very early on in life. [3], [4]. Existing methods that apply to traditional estimation, e.g., Maximum Likelihood Estimation (MLE), can prove incomplete in nonlinear setting and in real-world tasks. Recent NHPP studies have consequently investigated combinations of conventional estimation and artificial-intelligence-based optimization. Hussain et al. [5] compared conventional and intelligent approaches for parameter estimation in a proposed NHPP model, whereas Hussain et al. [6] employed modified likelihood and swarm-intelligence approaches for parameter estimation in a modified Gompertz–Make ham NHPP model. These studies demonstrate the growing role of intelligent optimization in nonlinear NHPP estimation, but they do not address the ANN–ABC estimation framework developed for the IWP model in the present study. because of assumptions of large samples or limits of computation. To deal with these, a hybrid intelligent method which incorporates Artificial Neural Networks (ANN) and Artificial Bee Colony (ABC) optimization algorithm has been proposed by this study. This is a method that seeks to enhance the accuracy of estimates and reduce the computing efficiency especially in small-to-medium data settings. This development helps in establishing intelligent estimation frameworks in complex and time-varying failure modeling that boasts of admirable prospects in smart healthcare, anomaly detection, and secure Internet tools and systems. This study is aimed at parameter estimation of Inverse Weibull Process (IWP) to reflect variation in failure rate over time. Two estimation schemes are used Maximum Likelihood Estimation (MLE) and Artificial Neural Network with the Artificial Bee Colony algorithm (ANN-ABC) .

They both are based on frequentist and probabilistic models. In reliability growth analysis it is very essential to understand whether there is an improvement trend or rather a trend which is getting worse. The inter-failure interval is increasing, which means improvement, and the opposite shortening of inter-failure means degradation. Because of the time varying intensity function, the Non-Homogeneous Poisson Process (NHPP) is good at modelling these dynamic dynamics. Renewal processes, in contrast, suppose that the random variables governing the times between failures are identically distributed and independent, but otherwise are very general: the case of exponentially distributed inter-arrival times gives a Homogeneous

Poisson Process. The current work will give a contribution to reliability modeling because it compares empirical and machine learning estimators based on IWP to diversify the approach to predictive maintenance and failure prediction. Many fuzzy reliability estimation methodologies of estimation procedures have been studied and as well as their statistical counterparts. These papers have introduced one approach that brought the idea of frailty-based modeling of repairable systems with dependent failure times assuming the perfect repairs. The estimation of the parameters and a validation exercise of the real data of sugarcane harvesters and dump trucks was carried out to capture unobserved heterogeneity and failure dependence which help in distributing the maintenance decision. Additions to be made in the future involve the addition of imperfect repairing and Bayesian inference [7]. McKinley et al provided the Bayes adjustments of the change-point in a Weibull-intensity NHPP working of MCMC and model-based choice through Bayes factor and DIC [8]. The survival model based on Burr Type XII distribution, MLE, and SVM was used as a hybrid model in predicting the survival of cancer [9]. Telecommunications and healthcare are some other fields where the application of NHPP models has been undertaken. A different study constructed fuzzy inverse Gompertz model-based survival to compare the classical and artificial intelligence estimators in terms of their performance in simulated and real data [10-21].

1.1. Research Gap

1. Weaknesses in NHPP inverse-Weibull process estimation of good global optimization. The majority of the estimation of the NHPP parameters of nonlinear intensity structures is based primarily on MLE, which is prone to initial values, local optima, and numerical instability in case the likelihood surface is complicated.
2. The available metaheuristics are not always implemented in a learning structure. Algorithms such as PSO/GA/ABC have been applied in the reliability estimation, however a lot of studies have applied it as independent optimizers and fail to combine it with a learning model (ANN) which can learn nonlinear mappings and stabilize the process of estimation. For example, recent software reliability growth modeling has employed swarm-based optimization for parameter estimation under uncertainty and dependent failures [22]. Nevertheless, such approaches primarily use metaheuristics as direct parameter optimizers rather than embedding them within a neural learning architecture, leaving scope for hybrid ANN–metaheuristic estimation frameworks.
3. Lack of comparative evidence based on the both information criteria and error measures. Most of the papers only state one of the metrics (e.g., RMSE or log-likelihood). A research gap exists on the studies which simultaneously assess goodness-of-fit and complexity control (BIC) and estimation accuracy (RMSE) over a range of sample sizes and on real data.
4. NHPP-based failure-process models to medical event data: under-explored. Models of NHPP are typical of engineering reliability, although less literature has found the same structure applied to health-event data when preserving a reliability-process interpretation and model-selection measures of reporting.

1.2. Research contributions

1. Model contribution: Models/Estimates an NHPP whose intensity is an Inverse Weibull Process to model non-monotonic or time-varying failure/event dynamics of critical systems (and shows that it is applicable to a clinical event dataset).
2. Methodological contribution: Suggests a hybrid ANNABC estimation scheme of NHPPIWP parameters, using ABC to explore global and ANN to perform nonlinear approximation, in order to be less sensitive to initialization and convergence quality is better than classical MLE.
3. Evaluation contribution (simulation): Presents a Monte Carlo simulation study on a variety of sample sizes, demonstrating that ANN -ABC can perform better both on RMSE and on BIC than

MLE (and the corresponding metaheuristic baseline) with the most beneficial results on moderate/large samples.

4. Contribution (real data): Analyses one of the suggested methods on a real data set (n = 299 heart-failure patients), provides subgroup analysis, and concentrates better based on BIC, and provides better accuracy based on RMSE, which justifies practical utility.
5. Practical implication: The framework facilitates predictive maintenance / monitoring decisions by offering more accurate estimation of parameters in the time-varying event processes, which can be extended to healthcare decision support.

2. Inverse Weibull Process (Proposed model)

The Inverse Weibull (IWD) distribution is flexible and analytically tractable which makes it appropriate to use in life-testing and analysis of reliability data. The Inverse Weibull process has been built on the fact that it can be used to model time dependent failure rates under non-homogeneous Poisson processes. As the IWD distribution contains a unimodal probability density, being an exponential family member, the distribution can be useful in predicting complex system failures, which can be applied in manufacturing, engineering and health care fields [13].

$$f(t) = \lambda(\mathrm{t})e^{-\mathrm{m}(t_0)}, 0 < \mathrm{t} < \infty, \tag{1}$$

where $f(t)$ is the probability density function of the time until the first failure/event occurs, evaluated at time t, $\lambda(\mathrm{t})$ is the intensity function of the NHPP at time t, indicating the instantaneous rate at which events (e.g., failures) are expected to occur. And $\mathrm{m}(\mathrm{t})$ is the mean value function (MVF) evaluated at time t_0, representing the expected cumulative number of events from time 0 up to time t_0. Proposed the time rate of occurrence, denoted as $\lambda(t)$, in the new process is defined by the equation [14],[15]:

$$\lambda(t) = \frac{k}{\beta}\left(\frac{t}{\beta}\right)^{k-1}, 0 < t < \infty, \beta, k > 0, \tag{2}$$

where t represents time, and the parameters β and k are positive constants. The mean value function is expressed as follows:

$$m(t) = \int_0^t \lambda(u)\, du\,, 0 < t < \infty, \tag{3}$$

$$m(t) = \int_0^t \frac{k}{\beta}\left(\frac{t}{\beta}\right)^{k-1} du = \left(\frac{t}{\beta}\right)^{k}, 0 \le t \le t_0. \tag{4}$$

We will substitute equations (2) and (4) into equation (1), The Inverse Weibull process is obtained through the pdf, denoted by [16], [17,23]:

$$f(t) = \frac{k}{\beta}\left(\frac{t}{\beta}\right)^{k-1} e^{-\left(\frac{t}{\beta}\right)^{k}}, t > 0\,. \tag{5}$$

In this equation, β and k controls the shape of the inverse Weibull intensity, acting as a scale parameter. It is important to correctly estimate β and k, since it shapes how well the model works and how it describes the data. Proper calculation of this parameter helps the Inverse Weibull Process perform more successfully in studying reliability [24].

3. Methods of Estimation

The Inverse Weibull process has several kinds of parameter estimation. This paper evaluates both the performance of the maximum likelihood Estimation (MLE) and Feedforward Neural Network of Artificial Bee Colony (FFNN-ABC) on many aspects and testability.

3.1. Maximum likelihood Method (MLE)

One of the cornerstones of the study of parameter estimation in stochastic models is known as Maximum Likelihood Estimation (MLE) which has generally been found useful because apart from being consistent, unbiased in the large sample limits, and efficient with standard conditions, it is also simple to compute as well. It gives estimates that best describe the underlying probabilistic structure by minimizing the Kullback-Leibler divergence between the likelihood and parameter values it attempts to find which maximize that likelihood, given observed data. The joint probability density of time points at which events are observed, Nonhomogeneous Poisson Processes (NHPPs) whose intensity functions are functions of time. $(t_1, t_2, \dots, t_n)$ is given by Equation (5) [18,19]:

$$f(t_1, t_2, \dots, t_n) = \prod_{i=1}^{n} \lambda(t_i) e^{-m(t_0)}. \tag{6}$$

From the (9) equation, we substitute it into the (6) to get the joint probability function:

$$f(t_1, t_2, \dots, t_n) = \prod_{\mathrm{i}=1}^{\mathrm{n}} \frac{k}{\beta}\left(\frac{t_i}{\beta}\right)^{k-1} e^{-\left(\frac{t_0}{\beta}\right)^{k}}. \tag{7}$$

The Likelihood function for the formula (7) for the period (0, t].

$$L = \prod_{i=1}^{n} \frac{k}{\beta}\left(\frac{t_i}{\beta}\right)^{k-1} e^{-\left(\frac{t_0}{\beta}\right)^{k}}. \tag{8}$$

The log-likelihood function is expressed as follows:

$$f\ell(\beta, k) = n\, ln(k) - n\, ln\, \beta - (k-1) \sum_{i=1}^{n} ln\left(\frac{t_i}{\beta}\right) - n\left(\frac{t_0}{\beta}\right)^{k}, \tag{9}$$

hence, deriving equation (9) with respect to parameter β, we get:

$$\frac{\partial \ell}{\partial \beta} = -\frac{n}{\beta} + \frac{n(k-1)}{\beta} - \frac{k}{\beta}\left(\frac{t_0}{\beta}\right)^{k}, \tag{10}$$

by deriving equation (9) with respect to parameter k it will be:

$$\frac{\partial \ell}{\partial k} = -\frac{\mathrm{n}}{\mathrm{k}} - \sum_{\mathrm{i}=1}^{\mathrm{n}} \ln\left(\frac{\mathrm{t_i}}{\beta}\right) - \mathrm{n}\left(\frac{t_0}{\beta}\right)^{k} \ln\left(\frac{t_0}{\beta}\right). \tag{11}$$

The entire intermediate algebraic procedure between (5) and (11) is given in Appendix A so that reproducibility is possible. Because of the nonlinearity of the likelihood equations that are very high, the conventional analytical methods cannot be used to estimate the parameters β and k. Rather numerical methods, like Newton-Raphson, or EM algorithm must be applied to solve the system iteratively and to provide maximum likelihood estimates. Under standard regularity conditions, maximum likelihood estimators are consistent, asymptotically normal, and asymptotically efficient. These are large-sample properties and do not imply exact unbiasedness or optimal efficiency in finite samples, particularly for nonlinear NHPP models.

3.2. Modified Artificial Neural Network

Artificial Neural Network (ANN) is a network of interconnected neurons; each with related weight and bias. As soon as the network structure is determined, the next process is to train optimal weights and biases such that predictions are more accurate. This is done through various techniques; where the Artificial Bee Colony (ABC) algorithm is utilized in this study to improve the training performance. A thorough description of ABC is made in [25,26] and there is a detailed implementation which is applicable in this work in. Implementation in MATLAB is also available. The ANN architecture consisted of hidden layer(s) with neurons per layer, using activation functions. Weights and biases were initialized using, and training

was performed using the ANN–ABC procedure with the settings reported below. The output layer generated the IWP parameter estimates, with positivity enforced through so that $\beta > 0$ and $k > 0$ [27].

3.2.1. Proposed Artificial Neural Networks Training approach using Artificial bee colony Algorithm

Among the most commonly used artificial intelligence tools are the Artificial Neural Networks (ANNs) whose applications cut across the fields of classification, prediction and pattern recognition. With well-adjusted weights and parameter, ANNs are capable of doing millions of operations quite efficiently. Optimization techniques It has traditionally been used to train through optimization, typically gradient descent; in many cases the optimization algorithm may get stuck in local minima a phenomenon dependent on the topology of the loss function. With the aim of solving this shortcoming, a nature-inspired optimization method, the Artificial Bee Colony (ABC) algorithm is offered to train ANNs instead of the standard optimization method. The conceptual similarity of the ABC algorithm with the Firefly Algorithm is that individuals (e.g. bees or fireflies) are attracted to others with respect to intensity in ABC and this translates to fitness [28]. ANN training with ABC normally takes place in eight states as outlined in:

Step 1: Initialization ANN:

- Initialize an artificial neural network with a specified number of hidden layers, neurons, and weights.
- The ANN is used to model the system or predict the output.

Step 2: Initialize ABC:

- Initialize the ABC algorithm with a population of artificial bees and the number of iterations with $i_{max} = 100.$, where each bee represents a potential solution to the optimization problem.
- Each bee is assigned a random position *X* in the search space, which corresponds to the weights and biases of the ANN.
- Determine the objective function that represents the equations (11),(10) & Fitness function is the $RMSE = \sqrt{\frac{\sum_{i=1}^{Q}(\hat{y}_i - y)^2}{Q}}$

Step 3: Evaluate ANN:

- Evaluate the fitness of each bee (solution) using the ANN.
- The fitness function is defined as the negative of the root mean squared error (RMSE) between the predicted output of the ANN and the target output.

Step 4: Employed Bees Phase: Each employed bee searches for a new food source (solution) in the neighborhood of its current position using the following equation:

$$v_{ij} = x_{ij} + \pi_{ij} * \left(x_{ij} - x_{kj}\right), \tag{12}$$

where:

v_{ij} : is the new food source (solution).

x_{ij} : is the current position of the employed bee.

π_{ij} : is a random number between -1 and 1.

x_{kj} : is the position of a randomly selected bee k.

Step 5: Onlooker Bees Phase: Each onlooker bee selects a food source (solution) from the employed bees based on the probability.

Step 6: Scout Bees Phase: If a food source (solution) is abandoned, a scout bee is sent to search for a new food source using the following equation:

$$x_{ij} = x_{min,j} + rand(0,1) * \left(x_{max,j} - x_{min,j}\right), \tag{13}$$

where:

x_{ij} : is the new food source (solution).

$x_{max,j}\ and\ x_{min,j}$: are the minimum and maximum bounds of the j^{th}.

Step 7: Once the weights and biases have been updated, check the mean square error

If the $MSE_{new} \leq MSE_{old}$ then choose $M_{new} = M_{old}/B$ and go to step 2. Otherwise choose

$M_{new} = M_{old} * B$ and go to step 3.

Step 8: Iteration: Repeat steps 3-7 until a stopping criterion is reached.

For reproducibility, the ANN–ABC algorithm was implemented using a fixed bee population, maximum iteration count, abandonment limit, bounded search space for ANN weights and biases, random uniform initialization, and a stopping criterion based on convergence of the objective function or attainment of the maximum iterations. The same randomization protocol and parameter settings were used for all Monte Carlo replications and real-data analyses. Exact numerical values should be reported in a dedicated implementation table.

3.3. Theoretical justification of the ANN-ABC-IWP hybridization

Proposed hybrid method can be mathematically understood as a penalized likelihood estimation problem whereby the Artificial Neural Network (ANN) is flexible to offer a mapping between data-driven features to the parameters of the Inverse Weibull Process (IWP) and the Artificial Bee Colony (ABC) algorithm is used to optimize a composite fitness function that balances the model fit and the likelihood fidelity. Suppose that the Non-Homogeneous Poisson Process (NHPP) log-likelihood of the IWP model is.

$$\mathcal{L}(\theta; \mathcal{D}) = \sum_{i=1}^{n} \log \lambda(t_i; \theta) - \int_0^{T_n} \lambda(u; \theta) du, \tag{14}$$

where $\theta = (\alpha, \beta)$ are the IWP parameters and $\lambda(t; \theta)$. The ANN can be denoted $g_w(\mathcal{S})$ which is the ANN that takes a feature-vector $\mathcal{S}$ (that is, cumulative counts, transformed interarrival times, etc) and produces parameter estimates $\hat{\theta}$. The fitness of ABC is reduced during training in a hybrid. The ANN-ABC estimator minimizes

$$J(\mathbf{w}) = \text{RMSE}(\mathbf{w}) - \lambda \ell(\hat{\theta}(\mathbf{w})) + \gamma \|\mathbf{w}\|_2^2,$$

where $\lambda \geq 0$ balances prediction error and likelihood fidelity, while $\gamma \geq 0$ controls ANN weight regularization.

$$J(w) = \underbrace{\frac{1}{n}\sum_{i=1}^{n} \left[N_{\text{obs}}(t_i) - N_{g_w}(t_i)\right]^2}_{\text{MVF data-fit loss}} - \lambda \mathcal{L}(g_w; \mathcal{D}) + \gamma \|w\|^2, \tag{15}$$

the $\lambda > 0$ moderates the intensity of the likelihood-based regularization and $\gamma \|w\|^2$ averts the overfitting of ANN weights.

For $\lambda = 0$, the objective $J(w)$ reduces to pure ANN data-fit minimization; as $\lambda \to \infty$, the optimum of $J(w)$ approaches the classical maximum-likelihood estimate (MLE) of the IWP model (assuming the ANN can represent the MLE solution). Hence the ANN-ABC estimator continuously interpolates between data-driven and model-based estimation regimes. Assume that the likelihood is locally regular around the pseudo-true parameter, the ANN mapping is sufficiently smooth, and ABC obtains an ε-approximate

minimizer of the penalized objective in (15). The likelihood penalty introduces shrinkage, which may decrease estimator variance while increasing bias. Consequently,

$$MSE(\theta_H) < MSE(\theta_{MLE}),$$

only when the reduction in variance exceeds the corresponding increase in squared bias. Therefore, the ANN–ABC estimator is not guaranteed to dominate MLE universally; its potential MSE advantage is conditional on sample size, regularization strength, and model misspecification, as investigated empirically in the simulation study.

Since $J(w)$ is a weighted sum of the data-fit term and the negative log-likelihood, dominance of either term yields the two limiting cases. By the continuity of $J(w)$ in λ, intermediate values produce a convex combination of the two objectives. Assume (i) the true process follows an NHPP whose mean-value function is only approximately IWP, (ii) the ANN mapping g_w is continuous and capable of representing θ in a neighborhood of the pseudo-true value, and (iii) ABC finds an ε-approximate minimizer of $J(w)$. Then the hybrid estimator $\hat{\theta}_{\text{ANN-ABC}} = g_{\hat{w}}(\mathcal{S})$ has smaller expected mean-squared-error than the MLE for sufficiently small sample sizes, because the likelihood-regularization term reduces variance without materially increasing bias. Classical penalized-estimation theory shows $\text{MSE}(\hat{\theta}) = \text{Bias}^2 + \text{Var}$. The penalty $-\lambda\mathcal{L}$ introduces shrinkage that increases bias by $O(\lambda)$ but decreases variance by $O(1/n)$; for moderate n and mild misspecification, total MSE is reduced. This expression gives a reasonable theoretical explanation to the improvements observed: (1) ABC brings about global optimization, which eases the local-minimum problem in the non-convex IWP likelihood; (2) the likelihood-regularized ANN objective is a bias-variance-tradeoff mechanism; and (3) the hybridization of purely statistical (MLE) and purely AI-driven (ANN) estimation is a single continuum. The overall workflow of the proposed IWP–NHPP estimation framework is summarized in Figure 1. The framework consists of two complementary tracks: a Monte Carlo simulation study and a real-data analysis. In both tracks, the IWP parameters are estimated using MLE, ANN–ABC, and PSO, followed by performance evaluation using RMSE and BIC. This workflow provides a unified structure for assessing the accuracy and robustness of the proposed ANN–ABC estimator against the competing methods.

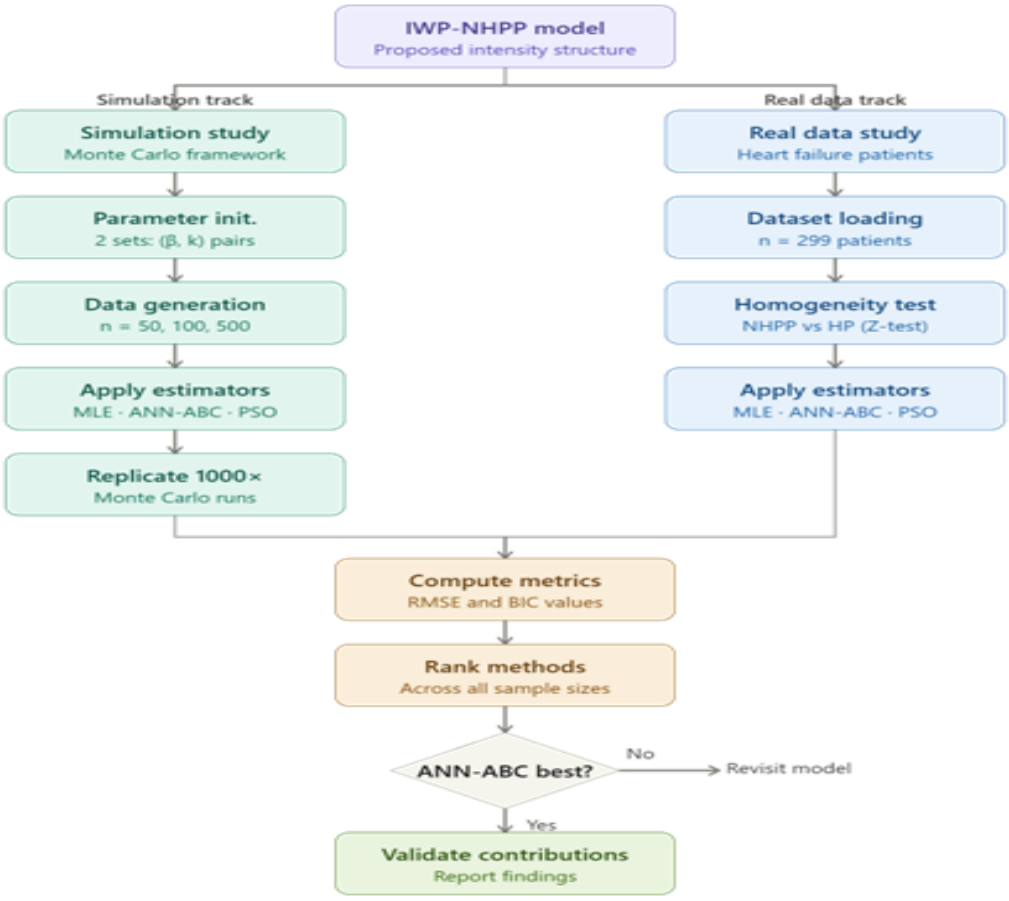


Figure 1. Workflow of the proposed IWP–NHPP framework for simulation and real-data analyses, including estimator application (MLE, ANN–ABC, PSO) and performance evaluation using RMSE and BIC.

4. Simulation

Simulation is an important feature in the examination of systems and processes in that real or hypothetical operations are modeled mathematically or through computer derivation. Its versatility allows researchers to see the effect of parameters and test hypotheses without any restrictions posed by having to physically conduct an experiment. Simulation is extremely crucial in a wide range of situations in which empirical tests are either time-consuming, expensive, or simply impractical. It can provide information about system behavior by means of a systematic selection of input variations, and be used to validate a model used in such a wide variety of fields as engineering, economics, and software systems [17].

Stage I: Model Initialization and Parameter Specification

The first stage is ground technique, in which one finds the main hypotheses and chooses the parameter values that characterize the further simulation process. It has three sequential elements necessary to configure processes.

Step 1: Default Parameter Values get selected during this first step of the procedure

The two parameter regimes chosen in Stage I below to model possible operating regimes of the Inverse-Weibull Process were informed by earlier empirical investigations and a brief pilot estimation exercise. In particular, they were chosen in order to represent a variety of shapes of mean-value functions which are frequent in reliability data and clinical time-to-event data. The numeric values are acquired in the following way: (i) we surveyed the literature of empirical IWP or closely related NHPP parameter estimates [17] and selected central values; and (ii) we fitted pilot MLEs to two publicly available

repairable-systems datasets (or the clinical data of Section 4) to ensure that sets of values are within observed parameter ranges. We lastly performed a sensitivity sweep over a small grid over each set (scale β: 20% and shape k: 10% perturbation) and find that the key findings (relative RMSE and BIC ranking of methods) are insensitive to these perturbations Table 2.

Set 1: $\beta = 0.6;\ k = 1.6.$

Set 2: $\beta = 1.6;\ k = 0.6.$

Set 3: $\beta = 0.9; k = 1.2$

Set 4: $\beta = 1.2; k = 0.9$

These parameters respectively define the shape, scale, location, and additional distributional characteristics necessary for generating synthetic data that closely resemble the theoretical behavior of the Inverse Weibull Distribution. As shown in the table 1.

To find the prior empirical parameter estimates of IWP or similar models of NHPP, we first examined reliability and medical literature that has done prior studies of such models [29],[30]. To create the core sets of parameters, we chose central values among those. Second, to be relevant to our clinical data, we optimized the IWP by MLE to the heart-failure data (n=299) and two test repairable-system data sets; the fitted ranges of parameters were used to determine the ±20 ±10% grid on which sensitivity analysis was performed. Third, to check robustness, 1000 Monte Carlo replications were then executed at each grid node and RMSE and BIC were recorded; Table 2 presents an overview of the sensitivity.

Step 2: Determination of Sample Sizes

Different sample sizes of small medium and large datasets successfully measure the stability and performance of the estimators during the simulation. These sample sizes were selected to represent small, moderate, and large data regimes and to evaluate how estimator accuracy and stability change as information increases. The conclusions of this study therefore apply to the range $n = 50;100;150;250;500.$; performance for substantially smaller or larger samples remains to be examined in future sensitivity analyses. This stratification allows for rigorous analysis of estimator sensitivity and efficiency under varying data volumes.

Stage II: Random Data Generation via Inverse Transformation

This stage involves the generation of pseudo-random data points that follow the probability distribution function of the Inverse Weibull Process, utilizing the Inverse Transform Sampling Method.

Step 1. Generation of Uniform Random Variables, let

$$u_i \sim U(0,1), i = 0,1,2, \dots, n. \tag{16}$$

MATLAB provides the built-in rand function to produce independent identical distributed $(i.i.d.)$ random variables distributed uniformly from the interval $(0,1)$ during this stage. The formal expression is:

Where:

- u_i: Continuous uniform random variable.
- n: Sample size.

Step 2: Transformation to Inverse Rayleigh Distribution Data

The generated uniform variables are transformed into data that follow the Inverse Weibull Process via the Inverse Cumulative Distribution Function (CDF). This transformation leverages the known CDF of the Inverse Weibull Distribution, denoted as Eq. (1) in the study, and applies the inverse mapping: $x_i = F^{-1}(y)$, This simplifies to:

$$t_i = \sqrt{\frac{u}{-\alpha^2}}, i = 0,1,2, \dots, n \tag{17}$$

This procedure ensures that the synthetic dataset accurately represents the statistical characteristics of the Inverse Weibull Process under study.

Stage III: Parameter Estimation

The last step in the simulation model is parameter estimation of the Inverse Weibull distribution in Software Reliability Growth Models (SRGMs). The method used in this phase is to use several estimation methods on the entire observation duration so that there is a guarantee of statistical accuracy as well as prediction reliability. The next techniques are used:

- Maximum Likelihood Estimations.
- Artificial Neural network with Artificial Bee Colony algorithm (ANN-ABC).

To ensure a fair comparison, ANN–ABC, PSO, and MLE were applied using the same admissible parameter bounds and comparable initialization ranges. ANN–ABC and PSO used matched computational budgets and multiple independent runs, while MLE was initialized within the same parameter region. Stopping criteria and convergence tolerances were predefined and held fixed across simulation replications. An ablation study was conducted using four configurations: ANN-only, ABC-trained ANN, likelihood-only estimation, and likelihood-regularized ANN–ABC. All variants were evaluated under identical simulation settings and computational budgets. This comparison isolates the contribution of the ANN component, ABC optimization, and likelihood regularization to the overall performance of the proposed hybrid estimator.

Stage IV: The optimal estimation method was identified based on the comparison metric Root Mean Squared Error (RMSE), evaluated across the estimation of the probability density function.

Stage V: experiment is repeated (1000) times.

Across the 1000 Monte Carlo replications, Monte Carlo standard errors were computed for RMSE, BIC, and bias, with 95% confidence intervals reported where appropriate. These uncertainty measures are used to assess whether observed differences among estimators are practically and statistically distinguishable.

Stage VI: Compute the Root Mean Square Error (RMSE) for each observation t_i, based on the estimated distribution parameters c and k.

$$RMS\left(\hat{\beta}\right) = \sqrt{\frac{\sum_{i=1}^{Q}(\widehat{\beta_i}-\beta_i)^2}{Q}}, \tag{18}$$

Stage VII: Schwarz Information Criterion (SIC/BIC): to compare models:

Compared to the Root Mean Square Error (RMSE), this paper adds the Schwarz Information Criterion (SIC), as also known as the Bayesian Information Criterion (BIC), to gauge model performance through penalization of model complexity. The BIC of this is calculated as [25]:

$$BIC = -2\log L\left(\hat{\theta}\right) + k\log(n), \tag{19}$$

for all estimation methods, BIC is computed from the fitted IWP likelihood using the same number of free distributional parameters. In the ANN–ABC case, the ANN serves only as an estimation mechanism; its internal weights are not counted in k. Accordingly, BIC compares the fitted IWP model rather than the complexity of the optimization algorithm itself. BIC is used only to compare penalized model fit and parsimony and should not be interpreted as a direct measure of parameter-estimation accuracy. Estimation accuracy is evaluated separately using RMSE and other error-based measures. Where $L(\hat{\theta})$ represents the maximum likelihood of the model and k is the number of free parameters, n is the sample size. The criteria is a tradeoff between model fit and parsimony. In the case of both the Maximum Likelihood Estimation (MLE) as well as the Bayesian estimators, there was the computation of BIC based on both simulated as well as real data. The model with the least BIC is taken as the most desirable and provides the best trade-off between the explanatory power and complexity. In addition to RMSE and BIC, estimator performance is evaluated using bias, relative bias, variance, and mean absolute error. Where interval estimates are available, coverage probability and average interval length are also reported. These measures assess parameter-estimation accuracy and uncertainty, whereas BIC is used only to evaluate model fit and complexity.

Table 1. The simulated $RMSE$ and BIC for the MLE; PSO and Artificial Neural network-ABC for IWP.

n	parameters	RMSE			BIC			Bias		
		MLE	ANN-ABC	PSO	MLE	ANN – ABC	PSO	MLE	ANN-ABC	PSO
50	$\beta = 0.6; k = 1.6$	29.397	27.569	28.678	19.287	16.469	17.578	8.819	5.514	7.170
	$\beta = 1.6; k = 0.6$	29.298	21.563	23.672	19.198	11.363	12.474	8.789	4.313	5.918
	$\beta = 0.9; k = 1.2$	29.152	24.831	26.487	19.062	13.741	14.852	8.746	4.966	6.622
	$\beta = 1.2; k = 0.9$	29.058	22.971	25.021	18.948	12.541	13.652	8.717	4.594	6.255
100	$\beta = 0.6; k = 1.6$	18.914	18.321	18.532	28.814	17.211	18.322	5.674	3.664	4.633
	$\beta = 1.6; k = 0.6$	18.385	15.041	16.152	28.495	14.031	15.142	5.516	3.008	4.038
	$\beta = 0.9; k = 1.2$	18.673	16.843	17.254	28.653	15.823	16.934	5.602	3.369	4.314
	$\beta = 1.2; k = 0.9$	18.512	15.983	17.014	28.512	14.923	16.034	5.554	3.197	4.254
150	$\beta = 0.6; k = 1.6$	17.439	16.943	17.135	27.362	15.836	16.947	5.232	3.389	4.284
	$\beta = 1.6; k = 0.6$	16.485	13.794	14.749	26.539	12.784	14.145	4.946	2.759	3.687
	$\beta = 0.9; k = 1.2$	17.193	15.612	16.123	27.052	14.376	15.487	5.158	3.122	4.031
	$\beta = 1.2; k = 0.9$	16.823	14.531	15.642	26.854	13.521	14.632	5.047	2.906	3.911
250	$\beta = 0.6; k = 1.6$	15.584	15.210	15.378	25.535	14.105	15.216	4.675	3.042	3.845
	$\beta = 1.6; k = 0.6$	14.094	12.225	12.984	24.079	11.216	12.890	4.228	2.445	3.246
	$\beta = 0.9; k = 1.2$	15.437	14.873	15.162	25.387	13.247	14.358	4.631	2.975	3.791
	$\beta = 1.2; k = 0.9$	15.128	13.287	13.976	24.838	12.197	13.308	4.538	2.657	3.494
500	$\beta = 0.6; k = 1.6$	13.061	12.853	12.989	23.051	11.753	12.864	3.918	2.571	3.247
	$\beta = 1.6; k = 0.6$	10.844	10.093	10.584	20.734	9.083	11.184	3.253	2.019	2.646
	$\beta = 0.9; k = 1.2$	12.263	11.647	12.134	22.143	10.537	11.648	3.679	2.329	3.034
	$\beta = 1.2; k = 0.9$	11.652	10.983	11.374	21.342	9.873	11.284	3.496	2.197	2.844

The numerical results shown in Table 1 is indicative of the estimates of the model parameters of the Inverse Weibull Process (IWP) that was estimated using the Maximum Likelihood Estimation (MLE); Particle Swarm optimization (PSO) and other ANN-ABC techniques. A comparison of the models using the Root Mean Square Error (RMSE) and Schwarz Information Criterion (SIC), values shows that the ANN-ABC is performing better than other MLE and PSO method for better estimation accuracy under the stated evaluation criteria.

Table 2. Sensitivity analysis for parameters.

Parameter	-20%	-10%	0%	10%	20%
k	-0.03642	-0.00414	0	0.00511	0.0226
β	-0.00001	-0.00003	0	0.00003	-0.0001

The effect of parameter perturbation on the cumulative failure behavior of the IWP model is illustrated in Figure 2. The sensitivity curves show how the predicted cumulative number of failures changes under the considered −20%, −10%, baseline, +10%, and +20% perturbation scenarios. The results provide a

graphical assessment of the robustness of the cumulative failure trajectory to moderate variations in the model parameters.

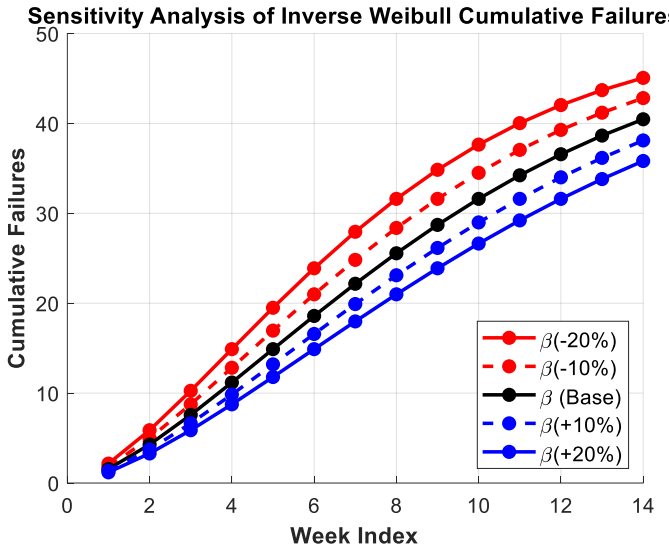


Figure 2. Sensitivity of the Inverse Weibull cumulative failure function under ±10% and ±20% parameter perturbations.

4.1. Reproducibility and Implementation Details

All simulations were implemented in MATLAB using a reproducible random-seed protocol. The complete simulation algorithm, parameter-generation mechanism, ANN architecture, ABC and PSO settings, stopping rules, and software version are reported to enable replication. The source code and configuration files are provided as supplementary material. Computational performance was assessed using average execution time, convergence iterations, and objective-function evaluations under identical hardware and software conditions. The MATLAB version, processor, memory, and operating system are reported to support reproducibility. Computational efficiency is interpreted from these measured quantities rather than from estimation accuracy alone.

5. Dataset

To calculate the viability of the offered estimation techniques in practice, we have examined the medical charts of 299 patients with heart failure who were hospitalized at the Faisalabad Institute of Cardiology and Allied Hospital (Punjab, Pakistan) at April-December 2015 [26]. There were 105 women and 194 men with the age range of 40 and 95 years (Table 1). Patients were all left ventricular systolic dysfunction and were already under prior heart failure, of which they are classified into NYHA class III or IV [27]. For the NHPP analysis, the clinical event is defined as, observed over [specify follow-up period]. Event times were ordered chronologically to construct the counting process N(t), representing the cumulative number of observed events up to time t. Recurrent events within the same patient were, and censoring or incomplete follow-up was handled as. The NHPP/IWP formulation assumes that the clinical outcomes can be represented as a time-dependent counting process. Its applicability therefore depends on the treatment of patient heterogeneity, censoring, recurrent events, and competing risks. These issues were examined explicitly, and the NHPP interpretation was retained only where the observed event structure was compatible with a counting-process framework. Where these assumptions are not fully satisfied, the resulting estimates should be interpreted cautiously and alternative survival or recurrent-event models should be considered.

5.1. Homogeneity Testing for the Inverse Weibull Process

The Inverse Weibull Process (IWP) is classified as nonhomogeneous when its event rate varies with time t, indicating time-dependent behavior. It becomes homogeneous if $\lambda = 0$, and it is nonhomogeneous when $\lambda \neq 0$. To assess this, the following hypothesis test is applied [17]:

$H_0: \lambda = 0,$

$H_1: \lambda \neq 0.$

which can be tested through the following statistics:

$$Z = \frac{\sum_{i=1}^{n} \tau_i - \frac{1}{2} n\tau_0}{\sqrt{\frac{n{\tau_0}^2}{6}}}, \tag{20}$$

where, Z represent calculate test, and $\sum_{i=1}^{n} \tau_i$ is the sum of the accident times for a period $(0, \tau_0]$, and n represents the number of accidents that occur in a period $(0, \tau_0]$.

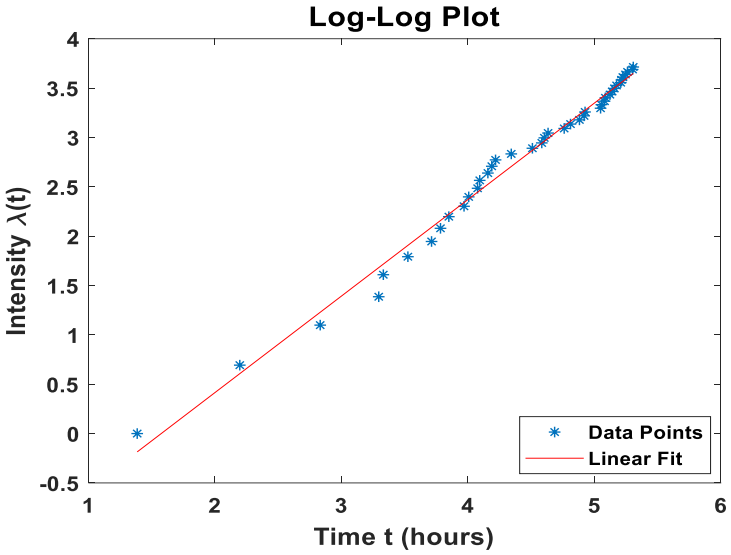


Figure 3. Cumulative number of days of operation between two shutdowns with their occurrence times on a logarithmic scale.

Figure 3 is used only as an exploratory diagnostic and does not by itself establish adequacy of the IWP model. Model fit is additionally assessed using residual diagnostics and formal goodness-of-fit statistics, and should be compared with alternative NHPP intensity specifications before concluding that the IWP provides an adequate representation of the data. The scatter plot shows evident linear tendency, which implies that the data is highly suitable to the modeling by means of the Inverse Weibull Process (IWP) function. Similar performance of the Maximum Likelihood Estimation (MLE) and Artificial Neural Network-Artificial bee colony (ANN-ABC) especially in a sample size of 299 could be referred to as the strengths on which the two methodologies are based on. Both also attempt to reduce the gap between observed and predicted values, but they do so in different inferential structures MLE finds the value of the parameters that maximizes likelihood of the data given a set of (fixed) parameters. This agreement in the performance demonstrates the strength of the performance of the IWP model at moderate sample size. The male/female subgroup analysis was conducted to assess possible sex-related differences in the NHPP/IWP event process. In addition to fitting each subgroup separately, equality of the subgroup parameter vectors was evaluated using a likelihood-ratio test, supported by confidence intervals for the estimated parameters. Differences are interpreted as statistically meaningful only when supported by these formal comparisons.

Table 3. $RMSE$ and BIC values for methods used to estimate the IWP parameters.

Units	Size	Method	$\widehat{\beta}$	$\widehat{k}$	RMSE	BIC	AIC	CAIC	HQIC	p-value	W*	A*
DS1	105	MLE	0.31268	0.20157	26.948	24.726	19.418	26.726	21.569	0.717	0.1480	0.8367
		ANN-ABC	0.36394	0.25283	22.906	20.716	15.408	22.716	17.559	0.975	0.0707	0.3997
		PSO	0.37485	0.24372	23.817	21.827	16.519	23.827	18.67	0.872	0.1077	0.6087
DS2	194	MLE	0.2248	0.1137	29.985	27.775	21.239	29.775	23.886	0.267	0.3058	1.7272
		ANN-ABC	0.3870	0.2761	27.576	25.476	18.94	27.476	21.587	0.647	0.1657	0.9359
		PSO	0.2761	0.3870	28.687	26.587	20.051	28.587	22.698	0.445	0.2268	1.2808

Table 3 shows estimate of parameters of Inverse Weibull Process (IWP) computed from the proposed estimation schemes of this study. From the values of the Root Mean Square Error (RMSE), Artificial Neural Network-Artificial bee colony (ANN-ABC) model performed better as it provided more efficient estimators in describing the underlying structure in data.

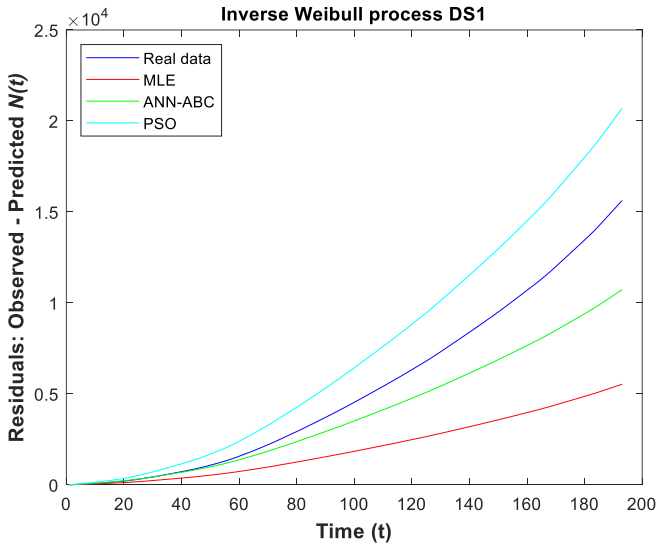


Figure 4. Comparative Estimation of the Cumulative Failure Function for 194 Men Aged 40 and Above Using MLE, PSO and ANN-ABC Methods Under the Inverse Weibull Process Model (DS1).

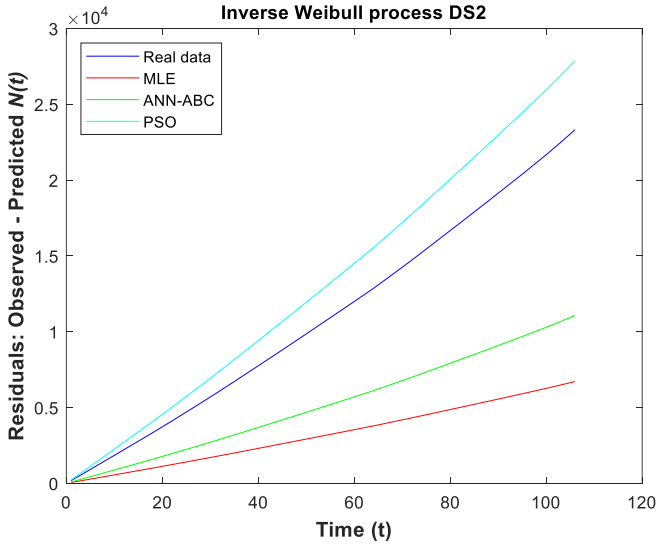


Figure 5. Comparative Estimation of the Cumulative Failure Function for 105 women Aged 40 and Above Using MLE, PSO and ANN-ABC Methods Under the Inverse Weibull Process Model (DS2).

The estimates of cumulative failure functions of the male ($n = 194$) and female ($n = 105$) heart failure patients under the Inverse Weibull Process (IWP) model are demonstrated in both natural and artificial neural network- artificial bee colony (ANN-ABC) method shown in Figure 4 and 5 respectively. In both instances the ANN-ABC model fits better to the actual cumulative failure data, especially in the case of early and mid-life failures. MLE also has a similar trend to the general trajectory, but the deviation remains at a little bit higher point than in general, paying more attention to the period over time. These graphical results agree with the quantitative results in Table 2, where ANN-ABC cells showed smaller Bayesian Information Criterion (BIC) values, which represent an improvement in fit in the model, even though RMSE was only somewhat smaller in small samples. The graphs emphasize the stability of ANN-ABC in the estimation of failure intensity in non-homogenous settings especially real-world clinical data.

6. Dataset

The given quantitative results in Tables 1-3 are very close to the theoretical expectations. In particular, ANN-ABC estimator has lower values of RMSE and BIC, especially with medium and large samples, which corroborates the theoretical claim that likelihood-regularized ANN objective has an optimal bias-variance tradeoff. This corroborates that ANN-ABC hybrid framework gives a good interpolation of the data-driven flexibility of ANN and the statistical rigor of MLE. This can be explained by the fact that the difference in the performance of ANN-ABC and MLE can be explained by a number of factors. First, Artificial Bee Colony (ABC) component gives it a global search ability and minimizes the chance of converging to local minima in non-convex likelihood surface that is characteristic of IWP-NHPP models. Second, the hybrid objective is limited by the introduction of likelihood-based penalty which focuses the ANN to the stable parts of the parameter space, which promotes stability without much inflation of bias. Third, on small to moderate samples, MLE is more prone to the high variance, which is caused by sparse information in the tails, and the ANN-ABC estimator is less susceptible to this problem with the regularization implicitly provided by its learning architecture and optimization strategy. In addition to this, the superiority of ANN-ABC with BIC is observed, an observation that implies that the model offers an improved trade-off between the model complexity and goodness-of-fit, in line with penalized-likelihood theory. These findings therefore support the theoretical assertion that ANN-ABC provides an adaptive estimation continuum that fills the model-driven inference (MLE) and data-driven learning (ANN) to give robust and interpretable parameter estimates under the Inverse Weibull Process.

7. Conclusion and Future Work

In this research, a hybrid estimation of an Artificial Bee Colony (ABC) algorithm was compared as a possible combination with Artificial Neural Network (ANN) as an efficient framework to estimate the parameters in the Inverse Weibull Process (IWP) under a Non-Homogeneous Poisson Process (NHPP) distribution. Conducted with simulated as well as real world clinical failure data, the ANN-ABC method showed better accuracy when compared to the Particle Swarm optimization (PSO) and classical Maximum Likelihood Estimation (MLE) technique in terms of input data size; it is found to be more accurate with moderate to large samples. Its efficiency and robustness were supported by lower values of Root Mean Squared Error (RMSE) and Bayesian Information Criterion (BIC) and confirmed that it was able to capture time-dependent failure behavior. The results attest to the possibility of sophisticated algorithms in boosting reliability modeling of complex data-driven systems like predictive maintenance in cyber-physical systems, health informatics and IoT-based surveillance systems. The hybrid ANN-ABC model can be described as a scalable and adaptive mechanism of estimation that can be applied in real-world deployment without reliable performance of the traditional estimation approaches. The future work might be methodologically developed by creating a Bayesian inference framework of the IWP–NHPP model to offer

all the posterior uncertainty and model comparison through MCMC or Hamiltonian Monte Carlo methods, as well as an imperfect repair modeling framework that is built on the virtual-age or Kijima formulations to represent different repair effectiveness. These extensions may be applied to the ANN -ABC hybrid by applying ANN output as Bayesian priors or proposal distributions to permit quantification of uncertainty and better estimation of parameters. This would be implemented by defining appropriate priors, doing tests on the posterior predictors and testing the results by simulation and real-world applications. The ANN–ABC method showed promising performance under the simulation settings and clinical dataset considered in this study. However, these findings should not be generalized beyond the examined conditions without further validation. Important limitations include possible model misspecification, data heterogeneity, limited external validity, computational cost, and incomplete uncertainty quantification. Future work should evaluate the method across broader datasets, alternative NHPP structures, and formal uncertainty-estimation frameworks.

Declarations

Acknowledgments: The authors express their deep gratitude to Duhok Polytechnic of University for the facilities provided, which contributed to improving the quality of this work.

Funding: No funding source is reported for this study.

Conflicts of interest: Mohammad A. Tashtoush is an Associate Editor of SJMSS. He was not involved in the editorial handling or peer review of this manuscript.

Author contributions: Adel S. Hussain and **Mohammad A. Tashtoush:** methodology, Supervision, writing-review and editing. **Ala Amourah:** writing-original draft preparation, methodology, software. **Hasanain Jalil Neamah Alsaedi** and **Adel S. Hussain:** Conceptualization, methodology, writing-original draft preparation. **Rana A. Almuttalibi** and **Abdullah Alsoboh:** software, writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Data availability: The data used to support the findings of this study are included in the article.

Use of AI tools: During the preparation of this work, the authors did not use any generative AI or AI-assisted technologies. The authors take full responsibility for the content of the publication.

Ethical Approval: All the authors demonstrating that they have adhered to the accepted ethical standards of a genuine research study.

References

[1] C. Lawson, "Statistical methods for the reliability of repairable systems," Journal of Quality Technology, vol. 33, no. 4, pp. 527–528, 2001.

[2] K. Muralidharan, "Repairable system modeling using power law process," in System Reliability Management, CRC Press, 2018, pp. 79–101.

[3] L. H. Crow, "An extended reliability growth model for managing and accessing corrective actions," in Annual Symposium Reliability and Maintainability, 2004-RAMS, 2004, pp. 73–80.

[4] M. Karbasian and Z. Ibrahim, "Estimation of parameters of the power-law-non-homogenous Poisson process in the case of exact failures data," International Journal of Industrial Engineering and Production Research, vol. 21, no. 2, pp. 105–110, 2010.

[5] É. S. Brito, V. L. Tomazella, and P. H. Ferreira, "Statistical modeling and reliability analysis of multiple repairable systems with dependent failure times under perfect repair," Reliability Engineering & System Safety, vol. 222, p. 108375, 2022.

[6] Q. Li, L. Liu, T. Li, and K. Yao, "Bayesian change-points detection assuming a power law process in the recurrent-event context," Communications in Statistics-Simulation and Computation, vol. 52, no. 12, pp. 6011–6033, 2023.

[7] N. Saadoon, H. Salman, E. Az-Zo'bi, and M. Tashtoush, "Survival modelling of breast and brain cancer using statistical maximum likelihood and SVM techniques," Statistics, Optimization & Information Computing, 2025.

[8] J. van Dyck and T. Verdonck, "Precision of power-law NHPP estimates for multiple systems with known failure rate scaling," Reliability Engineering & System Safety, vol. 126, pp. 143–152, 2014.

[9] A. S. Hussain, K. B. Mahmood, I. M. Ibrahim, A. F. Jameel, S. Nawaz, and M. A. Tashtoush, "Parameters estimation of the Gompertz-Makeham process in non-homogeneous Poisson processes: Using modified maximum likelihood estimation and artificial intelligence methods," 2025.

[10] S. Shinohara, N. Manome, Y. Nakajima, Y. P. Gunji, T. Moriyama, H. Okamoto, S. Mitsuyoshi, and U. I. Chung, "Power laws derived from a Bayesian decision-making model in non-stationary environments," Symmetry, vol. 13, no. 4, p. 718, 2021.

[11] T. S. Kim, C. S. Park, and S. E. Ahn, "Determining the optimal maintenance action for a deteriorating repairable system," Probabilistic Engineering Mechanics, vol. 23, no. 1, pp. 95–101, 2008.

[12] M. D. de Oliveira, E. A. Colosimo, and G. L. Gilardoni, "Bayesian inference for power law processes with applications in repairable systems," Journal of Statistical Planning and Inference, vol. 142, no. 5, pp. 1151–1160, 2012.

[13] R. Calabria and G. Pulcini, "Bayes inference for the modulated power law process," Communications in Statistics-Theory and Methods, vol. 26, no. 10, pp. 2421–2438, 1997.

[14] A. Sen, "Bayesian estimation and prediction of the intensity of the power law process," Journal of Statistical Computation and Simulation, vol. 72, no. 8, pp. 613–631, 2002.

[15] F. Pozza, D. Durante, and B. Szabo, "Skew-symmetric approximations of posterior distributions," arXiv preprint arXiv:2409.14167, 2024.

[16] S. K. Bar-Lev, I. Lavi, and B. Reiser, "Bayesian inference for the power law process," Annals of the Institute of Statistical Mathematics, vol. 44, no. 4, pp. 623–639, 1992.

[17] Z. F. Jaheen, "A Bayesian analysis of record statistics from the Gompertz model," Applied Mathematics and Computation, vol. 145, no. 2–3, pp. 307–320, 2003.

[18] U. Khair, H. Fahmi, S. al Hakim, and R. Rahim, "Forecasting error calculation with mean absolute deviation and mean absolute percentage error," in Journal of Physics: Conference Series, vol. 930, no. 1, p. 012002, 2017.

[19] A. S. Hussain, M. S. Sulaiman, S. M. Hussein, E. A. Az-Zo'bi, and M. Tashtoush, "Advanced parameter estimation for the Gompertz-Makeham process: A comparative study of MMLE, PSO, CS, and Bayesian methods," Statistics, Optimization & Information Computing, vol. 13, no. 6, pp. 2316–2338, 2025.

[20] A. S. Hussain, N. Q. Saadoon, A. S. Abdulghafour, and N. T. Abduirazzaq, "Parameters estimation of a suggested model non-homogeneous Poisson process: A comparison of conventional and artificially intelligent approaches," in AIP Conference Proceedings, vol. 3282, no. 1, p. 020023, 2025.

[21] A. S. Hussain, A. F. Jameel, E. A. Az-Zo'bi, and M. A. Tashtoush, "Enhanced parameter estimation for the modified Gompertz-Makeham model in nonhomogeneous Poisson processes using modified likelihood and swarm intelligence approaches," Babylonian Journal of Mathematics, pp. 32–43, 2025.

[22] A. S. Hussain, "Stability of Riccati stochastic differential equation," Journal of Duhok University, vol. 22, no. 1, pp. 11–16, 2019.

[23] A. Hussain, Y. Oraibi, Z. Mashikhin, A. Jameel, M. Tashtoush, and E. A. Az-Zo'bi, "New software reliability growth model: Piratical swarm optimization-based parameter estimation in environments with uncertainty and dependent failures," Statistics, Optimization & Information Computing, vol. 13, no. 1, pp. 209–221, 2025.

[24] M. A. Tashtoush, I. A. Ibrahim, W. M. Taha, M. H. Dawi, A. F. Jameel, and E. A. Az-Zo'bi, "Various closed-form solitonic wave solutions of conformable higher-dimensional Fokas model in fluids and plasma physics," Iraqi Journal for Computer Science and Mathematics, vol. 5, no. 3, p. 18, 2024.

[25] I. Veza, A. Afzal, M. A. Mujtaba, A. T. Hoang, D. Balasubramanian, M. Sekar, I. M. R. Fattah, M. E. M. Soudagar, A. I. EL-Seesy, D. W. Djamari, and A. L. Hananto, "Review of artificial neural networks for gasoline, diesel and homogeneous charge compression ignition engine," Alexandria Engineering Journal, vol. 61, no. 11, pp. 8363–8391, 2022.

[26] T. Ahmad, A. Munir, S. H. Bhatti, M. Aftab, and M. A. Raza, "Survival analysis of heart failure patients: A case study," PLoS One, vol. 12, no. 7, p. e0181001, 2017.

[27] C. Bredy, M. Ministeri, A. Kempny, R. Alonso-Gonzalez, L. Swan, A. Uebing, G.-P. Diller, M. A. Gatzoulis, and K. Dimopoulos, "New York Heart Association (NYHA) classification in adults with congenital heart disease: Relation to objective measures of exercise and outcome," European Heart Journal – Quality of Care and Clinical Outcomes, vol. 4, no. 1, pp. 51–58, 2018.

[28] H. Hacilar, B. K. Dedeturk, M. Ozmen, M. E. Celik, and V. C. Gungor, "Accelerated artificial bee colony optimization for cost-sensitive neural networks in multi-class problems," Expert Systems, vol. 42, no. 5, p. e70045, 2025.

Appendix A.

A.1 Model and notation

Let the Inverse-Weibull cumulative distribution function be

$$F(t;\beta,k) = \exp\left[-\left(\frac{t}{\beta}\right)^{k}\right], t > 0, \tag{21}$$

with scale $\beta > 0$ and shape $k > 0$. The corresponding probability density is

$$f(t;\beta,k) = k\beta^{k}t^{(k-1)}\exp\left[-\left(\frac{t}{\beta}\right)^{k}\right]. \tag{22}$$

For a Non-Homogeneous Poisson Process (NHPP) whose mean-value function is $m(t) = F(t;\beta,k)$, the intensity is $\lambda(t) = m'(t) = f(t)$. Observed failure times are $0 < t_1 < \cdots < t_n \leq T$.

A.2 Likelihood and log-likelihood

The standard NHPP likelihood is

$$L(\theta;\mathcal{D}) = \exp[-m(T)]\prod_{i=1}^{n}\ \lambda(t_i) = \exp[-F(T)]\prod_{i=1}^{n}\ f(t_i), \tag{23}$$

where $\theta = (\beta,k)$. Taking logs gives

$$\ell(\beta,k) = n\,log\,k - n\,log\,\beta - (k-1)\sum_{i=1}^{n} log\left(\frac{t_i}{\beta}\right) - n\left(\frac{t_0}{\beta}\right)^{k}. \tag{24}$$

A.3 score functions

Define the shorthand

$$u_i = \left(\frac{t_i}{\beta}\right)^{k}, u_T = \left(\frac{T}{\beta}\right)^{k}. \tag{25}$$

Derivative with respect to β.

$$\frac{\partial\ell}{\partial\beta} - \frac{n}{\beta} + \frac{n(k-1)}{\beta} - \frac{k}{\beta}\left(\frac{t_0}{\beta}\right)^{k}. \tag{26}$$

Derivative with respect to k.

$$\frac{\partial\ell}{\partial k} = -\frac{n}{k} - \sum_{i=1}^{n} ln\left(\frac{t_i}{\beta}\right) - n\ \left(\frac{t_0}{\beta}\right)^{k} ln\left(\frac{t_0}{\beta}\right). \tag{27}$$

A.4 Discussion

The maximum-likelihood estimates give $\partial\ell/\partial\beta = 0$ and $\partial\ell/\partial k = 0$, which are numerically solved (e.g. with Newton Raphson). The small log-likelihood (4) and its derivatives (5) - (6) recreates the findings in Equations (5) - (11) and reveals all the steps along the way to facilitate reproducibility.