Sigma Journal of Mathematical and Statistical Sciences (SJMSS)

Published by Altech Press • https://sjmss.altechpress.com

Vol. 1, No. 1 (2026), pp. 13–20

Research Article

Optical Soliton Solutions for the Conformable Nonlinear Hybrid-Lattice Equation Using the Generalized Kudryashov Method

Aysel Ramazanova [1], Hazal Yüksekkaya [2,*], Nasir Ali [3], Davron Aslonqulovich Juraev [4,5]

[1] *Department of Mathematics, Universität Duisburg-Essen, Essen, Germany,*

[2] *Department of Mathematics, Hakkari University, Hakkari, Turkey,*

[3] *Department of Mathematics, COMSATS University Islamabad, Vehari Campus, Vehari, Pakistan,*

[4] *Laboratory of Information Security and Cybersecurity, Azerbaijan University of Architecture and Construction, Baku AZ1073, Azerbaijan,*

[5] *Scientific Laboratory of Dynamical Systems and Their Applications, Institute of Mathematics, Uzbekistan Academy of Sciences, Tashkent 100174, Uzbekistan,*

*** Corresponding author:** hazalyuksekkaya@hakkari.edu.tr

Received: 15 August 2026 • Revised: 12 September 2026 • Accepted: 21 September 2026 • Published online: 30 September 2026

ABSTRACT

In this paper, we investigate the propagation of nonlinear waves in a discrete semi-infinite chain modeled by the conformable Hybrid-Lattice equation. By employing the continuum limit approximation, we transform the governing Differential-Difference Equation (DDE) into a nonlinear partial differential equation with fractional temporal evolution. We propose the application of the Generalized Kudryashov Method (GKM), a robust analytical technique that constructs solutions using exponential rational functions. This approach yields new exact solutions, including topological kink solitons and singular blow-up waves, which are essential for understanding energy transport and instability in nonlinear lattices. The impact of the fractional order α and the auxiliary base parameter A on the wave velocity and steepness is analyzed through comprehensive numerical simulations. The results demonstrate the efficiency of the GKM in solving fractional nonlinear evolution equations.

Keywords: Hybrid-Lattice equation; Conformable Calculus; Generalized Kudryashov Method; Optical Solitons; Discrete Systems; Singularity Analysis.

MSC 2020 Classification: 35C08; 35R11; 37K60

1. Introduction

Nonlinear Differential-Difference Equations (NDDEs) occupy a central position in mathematical physics, effectively modeling systems that possess discrete spatial structures but evolve in continuous time. These

models are ubiquitous in condensed matter physics, particularly in the study of atomic chains, molecular crystals, and biophysical systems such as DNA dynamics [1, 2]. Unlike purely continuous Partial Differential Equations (PDEs), NDDEs capture the intrinsic granularity of matter, leading to phenomena such as intrinsic localized modes and discrete breathers [3]. Recent studies have further expanded the analysis of such discrete systems to include Lotka-Volterra models [4] and integrable discrete hierarchies [5].

In recent decades, the concept of non-locality has been introduced into these lattice models through fractional calculus. While classical derivatives describe local changes, fractional derivatives incorporate the history of the system, a property known as memory effect [6]. This is crucial for modeling propagation in high-dispersive media [7] and complex magneto-acoustic systems [8]. To overcome the analytical difficulties associated with integral-based definitions (like Riemann-Liouville), Khalil et al. [9] proposed the Conformable Derivative, the physical interpretation of which has been extensively discussed by Zhao and Luo [10].

The search for exact traveling wave solutions for such complex systems requires sophisticated analytical methods. In the literature from 2015 to the present, several powerful schemes have emerged, including the Sine-Gordon expansion [11], the modified Khater method [12], and the functional variable method [13].

Concurrently, the study of optical solitons has garnered significant attention in broader nonlinear contexts. Recent investigations have successfully derived soliton solutions for birefringent fibers [14], nonlinear negative-index materials [15], and the nonlinear Schrodinger equation with conformable derivatives [16]. Furthermore, advanced rational function methods have been applied to resonant nonlinear Schrodinger equations to find novel wave structures [17].

In this paper, we utilize the GKM. Originally proposed to solve continuous PDEs, this method relies on a specific auxiliary ordinary differential equation (ODE) whose solutions are expressed in terms of exponential functions. This method is particularly effective in retrieving soliton solutions that decay exponentially at infinity, physically representing localized pulses [19, 20]. The method has been successfully adapted for various fractional evolution equations [21, 22]. We apply this method to the continuum limit of the Hybrid-Lattice equation to derive novel soliton solutions.

2. Preliminaries

2.1. Conformable Derivative

Given a function $f:[0,\infty)\to\mathbb{R}$, the conformable derivative of order α $(0<\alpha\leq 1)$ is defined as:

$$D_t^{\alpha}f(t)=\lim_{\epsilon\to 0}\frac{f\left(t+\epsilon t^{1-\alpha}\right)-f(t)}{\epsilon}. \tag{1}$$

A crucial property used in this work is the chain rule for composite functions [10]:

$$D_t^{\alpha}(f\circ g)(t)=t^{1-\alpha}g'(t)f'\big(g(t)\big). \tag{2}$$

2.2. Hybrid-Lattice Model

We consider the nonlinear Hybrid-Lattice equation with conformable time evolution that defined in [18]:

$$D_t^{\alpha}v_n(t)=\big(\lambda+\beta v_n(t)+\gamma v_n^2(t)\big)\big(v_{n+1}(t)-v_{n-1}(t)\big). \tag{3}$$

Here, $v_n(t)$ represents the displacement of the n-th particle. To analyze this system analytically, we apply the continuum limit approximation $(v_n(t)\to u(x,t))$. Using the Taylor series expansion up to $O(h^4)$:

$$v_{n+1}-v_{n-1}\approx 2hu_x+\frac{h^3}{3}u_{xxx}. \tag{4}$$

Substituting this into equation (3) yields the nonlinear PDE:

$$u_t^{(\alpha)} - (\lambda + \beta u + \gamma u^2)\left(2hu_x + \frac{h^3}{3}u_{xxx}\right) = 0. \tag{5}$$

3. Generalized Kudryashov Method

The GKM is a powerful algebraic technique for solving nonlinear evolution equations. The steps are defined as follows [19]:

Step 1: Transform the PDE into an ODE using the traveling wave variable $\xi = kx - \frac{c}{\alpha}t^{\alpha}$.

$$P(U, U', U'', \dots) = 0. \tag{6}$$

Step 2: Assume the solution $U(\xi)$ can be expressed as a polynomial in $Q(\xi)$:

$$U(\xi) = \sum_{i=0}^{N} a_i\, Q^i(\xi), \quad a_N \neq 0, \tag{7}$$

where $Q(\xi)$ satisfies the auxiliary equation:

$$Q'(\xi) = \left(Q^2(\xi) - Q(\xi)\right)\ln A, \quad A \neq 0,1. \tag{8}$$

The solution to this auxiliary equation is given by:

$$Q(\xi) = \frac{1}{1+DA^{\xi}}, \tag{9}$$

where D is an integration constant.

Step 3: Determine the integer N by balancing the highest order derivative and the nonlinear term in the ODE.

Step 4: Substitute the ansatz equation (7) and the derivatives of Q into the reduced ODE. Collect coefficients of Q^i to form a system of algebraic equations. Solving this system yields the parameters a_i, k, c.

4. Application and Exact Solutions

Applying the wave transformation $\xi = kx - \frac{c}{\alpha}t^{\alpha}$ to equation (5), we integrate once (setting the integration constant to zero) to obtain the balanced ODE typical for mKdV-like equations:

$$c_1 U'' + c_2 U^3 + c_3 U = 0, \tag{10}$$

where the coefficients c_1, c_2, c_3 are functions of the lattice parameters $\lambda, \beta, \gamma, h$.

Balancing U'' with U^3 in equation (10) yields $N + 2 = 3N \Longrightarrow N = 1$. Thus, the solution takes the form:

$$U(\xi) = a_0 + a_1 Q(\xi). \tag{11}$$

Computing the derivatives using $Q' = (Q^2 - Q)\ln A$:

$$U'' = a_1(\ln A)^2(2Q^3 - 3Q^2 + Q). \tag{12}$$

Substituting U, U^3, U'' into equation (10) and collecting powers of $Q(\xi)$, we get the following system of equations:

$$
\begin{aligned}
Q^3: &\quad 2c_1a_1(\ln A)^2 + c_2a_1^3 = 0,\\
Q^2: &\quad -3c_1a_1(\ln A)^2 + 3c_2a_0a_1^2 = 0,\\
Q^1: &\quad c_1a_1(\ln A)^2 + 3c_2a_0^2a_1 + c_3a_1 = 0,\\
Q^0: &\quad c_2a_0^3 + c_3a_0 = 0.
\end{aligned}
$$

Solving this system provides the relationships between a_0, a_1, c, k. For instance, from the Q^3 term:

$$a_1 = \pm\sqrt{-\frac{2c_1(\ln A)^2}{c_2}}. \tag{13}$$

Substituting these into equation (7) and using equation (9), we obtain the generalized Kudryashov solution:

$$u(x,t) = a_0 + a_1\left(\frac{1}{1+DA^{kx-\frac{c}{\alpha}t^\alpha}}\right). \tag{14}$$

5. Numerical Simulation and Discussion

In this section, we choose various parameter sets to investigate the physical dynamics of the derived solution (14). Before discussing each of the four cases separately, Table 1 below provides an overview of all exact solution regimes obtained using the Generalized Kudryashov Method. These regimes include the stable kink soliton, singular blow-up wave, the impact of the fractional order α on wave retardation, and the effect of the base parameter A on wave front steepness. The reader might use this table as a guide while thoroughly examining each case.

Table 1. Shows an overview of the conformable Hybrid-Lattice equation's exact-solution scenarios, parameter values, solution forms, and matching figures.

#	Case / Wave Type	Parameter Values	Solution Forms	Figure
1	Stable Kink-Topological soliton	$D = 1, A = e \approx 2.718,$ $\alpha = 0.98, a_0 = 0, a_1 = 1$	Logistic form, $u = \frac{1}{1+e^{kx-ct^\alpha}}$	Fig. 1 (3D spatiotemporal plot)
2	Singular Soliton - Blow-up wave	$D = -1, A = e \approx 2.718,$ $\alpha = 1.0$	Singularity at $A^{kx-(c/\alpha)t^\alpha} = 1$, i.e. $kx = (c/\alpha)t^\alpha$	Fig. 2 (3D singularity profile)
3a	Base-A Effect - Gradual wavefront	$A = e \approx 2.718, t = 1,$ $D = 1, \alpha = 1$	Baseline logistic, $\ln A = 1$	Fig. 3 (2D cross-section)
3b	Base-A Effect-Medium steepness	$A = 5, t = 1, D = 1, \alpha = 1$	Steeper transition, $\ln 5 \approx 1.6$	Fig. 3 (2D cross-section)
3c	Base-A Effect-High steepness	$A = 10, t = 1, D = 1,$ $\alpha = 1$	Sharpest transition, $\ln 10 \approx 2.3$	Fig. 3 (2D cross-section)
4a	Fractional Order $\alpha = 1$ (classical)	$\alpha = 1.0, t = 2, A = e,$ $D = 1$	$t^\alpha = t$, no memory effect	Fig. 4 (2D fractional comparison)
4b	Fractional Order $\alpha = 0.9$	$\alpha = 0.9, t = 2, A = e,$ $D = 1$	Mild retardation, $t^{0.9} < t$	Fig. 4 (2D fractional comparison)
4c	Fractional Order $\alpha = 0.75$	$\alpha = 0.75, t = 2, A = e,$ $D = 1$	Moderate retardation, $t^{0.75} \leq t$	Fig. 4 (2D fractional comparison)
4d	Fractional Order $\alpha = 0.6$	$\alpha = 0.6, t = 2, A = e,$ $D = 1$	Strong retardation, $t^{0.6} \leq t^{0.75}$	Fig. 4 (2D fractional comparison)

Case 1: Stable Kink Soliton Propagation

By choosing the integration constant $D = 1$ and the base $A = e$ (Euler's number), the solution simplifies to a logistic function, representing a topological kink. Kinks are crucial in lattice dynamics as they represent domain walls or transition states between equilibrium positions.

Figure 1 illustrates the 3D spatiotemporal evolution of this wave. The amplitude transitions smoothly from a_0 to $a_0 + a_1$. The wave travels without deformation, confirming the balance between nonlinearity and dispersion in the continuum limit of the Hybrid-Lattice.

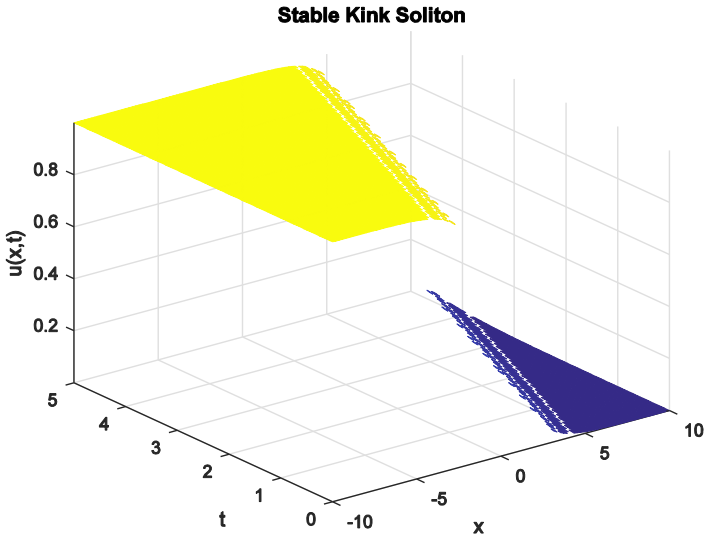


Figure 1. 3D evolution of the stable Kink Soliton at $a_0 = 0, a_1 = 1, D = 1, A = e$, and $\alpha = 0.98$.

Case 2: Singular Soliton Dynamics

Singular solutions, while often discarded in basic wave theory, are physically significant in the context of extreme events, such as wave breaking or plasma instability. By setting $D = -1$, the denominator in (4.5) approaches zero when $A^{kx-\frac{c}{\alpha}t^{\alpha}} = 1$, leading to a singularity.

Figure 2 displays this behavior. The solution exhibits a "blow-up" at specific spatiotemporal points. This corresponds to a region in the lattice where particle displacement exceeds the elastic limit of the model approximation.

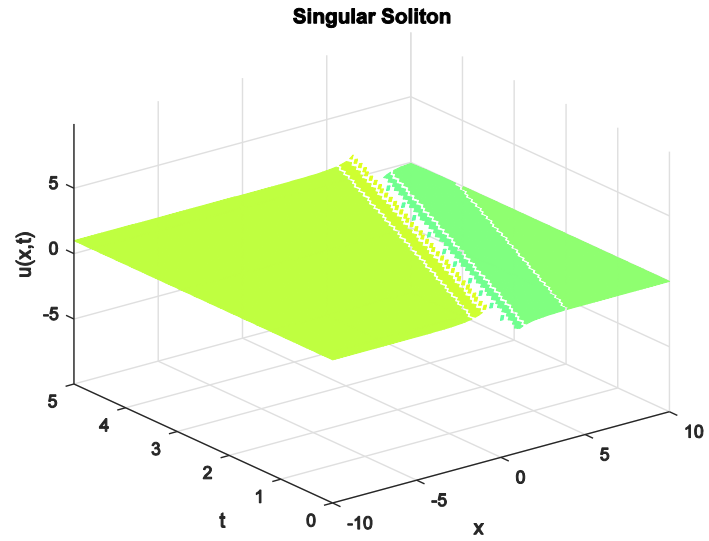


Figure 2. Profile of the Singular Soliton showing infinite discontinuity at $D = -1, A = e$, and $\alpha = 1.0$.

Case 3: Effect of Base Parameter A

The Generalized Kudryashov Method introduces the arbitrary base A. This parameter controls the steepness of the soliton interface. As shown in Figure 3, increasing A from e (approx 2.718) to 10 results in a much sharper transition (steeper gradient). This allows the method to model "shock-like" waves in the lattice without changing the fundamental velocity parameters.

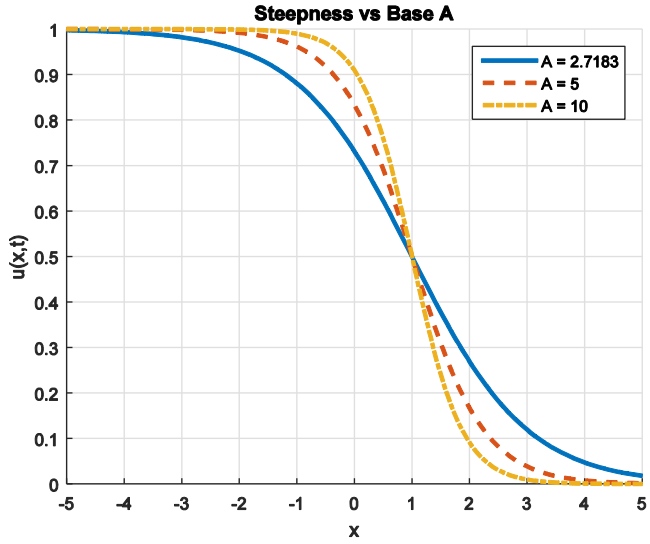


Figure 3. 2D cross-section comparison at $t = 1$ for different base values A. Higher A yields a steeper wave front.

Case 4: Influence of Fractional Order α

The conformable derivative order α encapsulates the memory effects of the material. As illustrated in Figure 4, as α decreases from 1 (standard derivative) to 0.75, the wave experiences a significant phase shift and retardation. This implies that in a "fractional" lattice, energy propagates slower due to non-local interactions between particles.

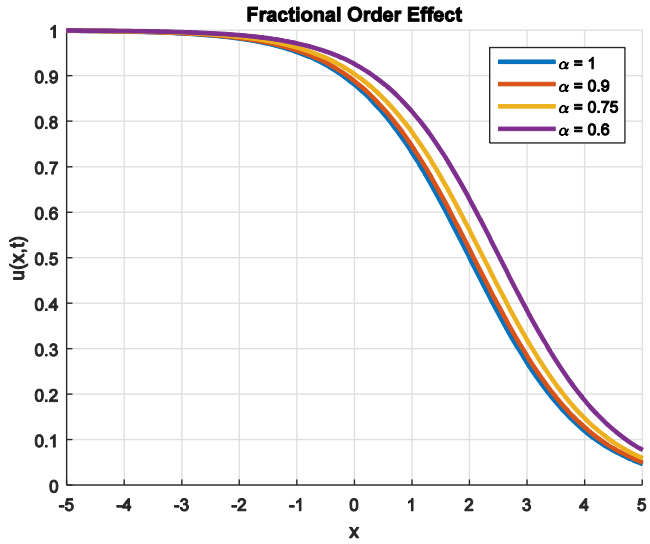


Figure 4. The impact of fractional order α on wave propagation at $t = 2$. Lower α indicates strong memory effects.

6. Conclusion

In this study, we successfully applied the Generalized Kudryashov Method to the conformable Hybrid-Lattice equation. We derived a general family of solutions dependent on the base parameter A. Numerical simulations revealed that the system supports both stable topological kink solitons and singular blow-up waves depending on the integration constants. Furthermore, we demonstrated that the fractional order α acts as a retardation parameter, slowing down the wave propagation, while the parameter A modulates the steepness of the wavefront. These results provide deeper insight into the energy transport mechanisms in nonlinear discrete systems with memory.

Declarations

Acknowledgments: Not Applicable.

Funding: This research received no external funding.

Conflicts of interest: The authors declare no conflicts of interest.

Author contributions: Aysel Ramazanova: methodology, Conceptualization, writing-review and editing. **Hazal Yüksekkaya:** software, investigation, writing-review and editing. **Nasir Ali**: validation, writing-original draft preparation, **Davron Aslonqulovich Juraev:** formal analysis, investigation, writing-original draft preparation. All authors have read and agreed to the published version of the manuscript.

Data availability: No data was used to support the results of this paper.

Use of AI tools: During the preparation of this work, the authors used Gemini only for language editing and grammar checking. After using this tool, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

References

[1] H. Kumar and S. El-Ganaini, "Traveling and localized solitary wave solutions of the nonlinear electrical transmission line model equation," *The European Physical Journal Plus*, vol. 135, no. 9, pp. 1–25, 2020.

[2] L. Wu, L. D. Xie, and J. F. Zhang, "Adomian decomposition method for nonlinear differential-difference equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 1, pp. 12–18, 2009.

[3] B. Tang and K. Deng, "Discrete breathers and modulational instability in a discrete Q4 nonlinear lattice with next-nearest-neighbor couplings," *Nonlinear Dynamics*, vol. 88, no. 4, pp. 2417–2426, 2017.

[4] A. Elsonbaty and A. A. Elsadany, "On discrete fractional-order Lotka–Volterra model based on the Caputo difference discrete operator," *Mathematical Sciences*, vol. 17, no. 1, pp. 67–79, 2023.

[5] V. S. Gerdjikov, B. Prinari, V. V. Pukhnachev, and M. D. Todorov, "Solitons, integrability, nonlinear waves: Theory and applications," *The European Physical Journal Plus*, vol. 136, no. 1, p. 41, 2021.

[6] A. Atangana and D. Baleanu, "New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model," *Thermal Science*, vol. 20, no. 2, pp. 763–769, 2016.

[7] S. Kumar, D. Kumar, and J. Singh, "Fractional modelling arising in unidirectional propagation of long waves in dispersive media," *Advances in Nonlinear Analysis*, vol. 5, no. 4, pp. 383–394, 2016.

[8] D. G. Prakasha, R. Saadeh, K. Kachhia, A. Qazza, and N. S. Malagi, "A new computational technique for analytic treatment of time-fractional nonlinear equations arising in magneto-acoustic waves," *Mathematical Problems in Engineering*, vol. 2023, no. 1, Art. no. 6229486, 2023.

[9] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, "A new definition of fractional derivative," *Journal of Computational and Applied Mathematics*, vol. 264, pp. 65–70, 2014.

[10] D. Zhao and M. Luo, "General conformable fractional derivative and its physical interpretation," *Calcolo*, vol. 54, no. 3, pp. 903–917, 2017.

[11] T. Iatkliang, S. Kaewta, N. M. Tuan, and S. Sirisubtawee, "Novel exact traveling wave solutions for nonlinear wave equations with Beta-derivatives via the sine-Gordon expansion method," *WSEAS Transactions on Mathematics*, vol. 22, pp. 432–450, 2023.

[12] M. Khater, S. Anwar, K. U. Tariq, and M. S. Mohamed, "Some optical soliton solutions to the perturbed nonlinear Schrödinger equation by modified Khater method," *AIP Advances*, vol. 11, no. 2, 2021.

[13] F. Al Basir and P. K. Roy, "Stability analysis and optimal control of a fractional order model for HIV infection," *WSEAS Transactions on Mathematics*, vol. 16, pp. 152–162, 2017.

[14] L. Tang, "Bifurcation analysis and multiple solitons in birefringent fibers with coupled Schrödinger–Hirota equation," *Chaos, Solitons & Fractals*, vol. 161, Art. no. 112383, 2022.

[15] P. P. Banerjee and G. Nehmetallah, "Linear and nonlinear propagation in negative index materials," *Journal of the Optical Society of America B*, vol. 23, no. 11, pp. 2348–2355, 2006.

[16] M. Alabedalhadi, M. Al-Smadi, S. Al-Omari, D. Baleanu, and S. Momani, "Structure of optical soliton solution for nonlinear resonant space-time Schrödinger equation in conformable sense with full nonlinearity term," *Physica Scripta*, vol. 95, no. 10, Art. no. 105215, 2020.

[17] M. Eslami and A. Neirameh, "Generalized exponential rational function for distinct types solutions to the conformable resonant Schrödinger's equation," *International Journal of Modern Physics B*, vol. 35, no. 30, Art. no. 2150306, 2021.

[18] M. Eslami, S. Heidari, S. A. Jedi Abduridha, and Y. Asghari, "Generalized exponential rational function method for solving nonlinear conformable time-fractional Hybrid-Lattice equation," *Optical and Quantum Electronics*, vol. 56, no. 5, p. 725, 2024.

[19] M. Kaplan, A. Bekir, and A. Akbulut, "A generalized Kudryashov method to some nonlinear evolution equations in mathematical physics," *Nonlinear Dynamics*, vol. 85, no. 4, pp. 2843–2850, 2016.

[20] A. Hussain, A. Jhangeer, N. Abbas, I. Khan, and E. S. M. Sherif, "Optical solitons of fractional complex Ginzburg–Landau equation with conformable, beta, and M-truncated derivatives: A comparative study," *Advances in Difference Equations*, vol. 2020, no. 1, p. 612, 2020.

[21] K. Hosseini, P. Mayeli, and R. Ansari, "Modified Kudryashov method for solving the conformable time-fractional Klein–Gordon equations," *Analysis and Mathematical Physics*, vol. 8, no. 1, pp. 15–26, 2018.

[22] A. C. Cevikel and E. Aksoy, "Soliton solutions of nonlinear fractional differential equations with their applications in mathematical physics," *Revista Mexicana de Física*, vol. 67, no. 3, pp. 422–428, 2021.