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Research Article

An Advanced Numerical Technique for Solving Nonlinear Fractional-Order Pseudo Hyperbolic Telegraph Equations

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ABSTRACT

In this manuscript, a novel theoretical framework and numerical method for solving nonlinear fractional-order pseudo hyperbolic telegraph equations (FPHTE) are presented. These formulas model the propagation of waves with memory effects in a variety of engineering and physical systems, including electromagnetic waves in plasmas, porous media, and viscoelastic materials. This equation is generated in a generalized form that is not dependent on any particular nonlinearity. This approach shows the correctness and dependability of the method for a variety of these equations by offering a methodical way to identify the mild solution and evaluate its stability. This study investigates how to obtain a numerical approximation of the nonlinear equations using the explicit finite difference approach. The authors develop a finite difference scheme, implement it in MATLAB, and employ various numerical experiments to assess its stability and accuracy. The results demonstrate that the explicit finite difference procedure is a reliable and efficient solution for the proposed framework. It accurately captures wave properties, including memory and nonlinearity effects, and is stable over a wide range of fractional orders. Maintaining numerical stability and accuracy requires careful consideration of the time step, spatial grid size, and fractional order.

Keywords: Nonlinear pseudo hyperbolic telegraph equations; Caputo fractional derivative; explicit finite difference technique; approximate results, stability.

MSC 2020: 35R11; 65M06; 65M12; 35L71.

1. Introduction

Fractional calculus has completely transformed the mathematical description of challenging physical systems that show anomalous diffusion, memory effects, and hereditary features [1, 2]. Fractional Partial

Differential Equations offer a more realistic representation of non-local dynamics in biological systems, signal processing, and viscoelastic materials than traditional integer-order models [3, 4].

Fractional models employ fractional calculus, a branch of classical calculus dealing with non-integer order derivatives and integrals. This method is especially appropriate for examining systems that exhibit memory and hereditary features, finding applications across physics, engineering, biology, and finance. By accounting for long-range dependencies and anomalous diffusion, fractional models provide a realistic depiction of complex processes. Consequently, they are valuable tools for understanding real-world phenomena that classical calculus may not capture well.

Nonlinear fractional-order pseudo hyperbolic telegraph equations (FPHTE) represent a class of mathematical models finding great application in various scientific and engineering disciplines. This equation extends the classical telegraph equation by incorporating fractional order derivatives, which establishes a non-local aspect to the system's dynamics. The existence of nonlinear terms adds complexity, allowing for the description of a various array of physical phenomena exhibiting nonlinearity. The pseudo hyperbolic nature of the equation implies a balance between hyperbolic and parabolic behavior, making it suitable for modeling wave-like processes with both dispersion and diffusion characteristics. These types of equations have gained importance due to their ability to describe complex systems with memory effects accurately. They are notable tools in fields such as electromagnetics, viscoelasticity, and fluid dynamics. In this context, understanding the properties, solutions, and numerical methods associated with fractional order nonlinear pseudo hyperbolic telegraph equations is of most important for advancing our understanding of real-world phenomena and innovative technologies.

The study of fractional-order partial differential equations, both linear and nonlinear, has recently garnered significant attention. Several academics have investigated the behavior of various types of these equations. For example, the authors of [1] devised a direct and efficient method for solving the nonlinear logistic equation of partial exponential functions. Their study highlighted the importance of using fractional-order Bessel and Legendre functions. Similarly, the authors of [2] provided a comprehensive analysis of numerical methods for solving fractional Riccati equations and discussed the effectiveness of implicit and explicit finite-difference methods in solving these equations.

The authors of [3] studied nonlinear reaction-diffusion equations with fractional derivatives of order ($0 < \alpha < 2$). A broad but limited area was used to study these equations. The authors of [4] presented a mathematical approach for solving a specific class of nonlinear partial differential equations. This approach proved effective by obtaining and graphically plotting accurate solutions. By incorporating the fractional operators of Caputo and Atangana-Baleanu Caputo according to the Caputo sense, the authors of [5] constructed a clear and convincing pattern for the outputs of KdV, mKdV, K (2,2), and K (3,3) models. To achieve this, they used the Jafari transform and the Adomian decomposition techniques.

The authors of [6] successfully applied a numerical method to solve fractional-order Fokker-Planck equations. Specifically, they defined fractional derivatives according to the Caputo sense using a homotopy perturbation method and an innovative iterative approach. According to [7], the authors used the Yang transform in conjunction with HPM to solve nonlinear fractional differential equations. The authors of [8] developed a Laplace-Adomian analysis method for studying fluid dynamics in relation to thermal heating. They presented an analytical solution to the nonlinear Swift-Hohenberg equations. To solve interrelated systems of nonlinear partial differential equations more efficiently using symmetry analysis and Adomian analysis, the authors of [9] developed a conformable fractional power series methodology.

The authors of [10] presented a unique approach to solving nonlinear FPDEs. This technique is known as the modified conformable Laplace-Sumudo double decomposition method. The authors developed a novel technique in [11] to predict accurate estimates of solutions to the hyperbolic telegraph equation with a

time-fractional component. This approach incorporates time-fractional derivatives in the sense of Caputo through the utilization of a Laplace transform operator and a fractional power series technique. The authors in [12] analyzed the characteristics of the (DCLT) and integrated it with a novel approach to solve a specific category of FPDEs.

The explicit finite difference method was employed by the author in [13] with respect to the FPHTPDE characterized by the Caputo fractional derivative. The strategy includes the development of a problem-specific first-order finite difference algorithm. Authors in [14] used a modified double Laplace transform approach to solve the PHTPDEs in Caputo sense. The authors of [15] presented a new approach to solving the fractional telegraph equations by using the Kronecker product to combine the Chebyshev polynomials with cubic radial basis function. The authors of [16] presented a novel operational matrix method that effectively solves time-fractional telegraph equations with Dirichlet boundary conditions by employing Fibonacci wavelets and block pulse functions. The authors of [17] proposed a novel hybrid strategy, called the He-Elzaki transform method, for solving the time-fractional telegraph equations in the electromagnetic field. To our knowledge, the stability and numerical solution of the nonlinear fractional-order pseudo hyperbolic telegraph equation in the Caputo sense have not been thoroughly explored using explicit finite difference methods. In an attempt to fill this gap in the literature, this work presents a comprehensive analysis of the accuracy and stability of this method. For this aim we consider the problem of solving nonlinear fractional order telegraph equation with given boundary conditions:

$$\begin{cases} w_{tt}(t,x) + {}_0^C D_t^\alpha w(t,x) + w^2(t,x) = \lambda w_{txx}(t,x) + w_{xx}(t,x) + f(t,x), \\ 0 < t < T, 0 < x < L, 0 \le \alpha < 1 \\ w(0,x) = \varphi(x), w_t(0,x) = \psi(x), \;\; 0 \le x \le L \\ w(t,0) = w(t,L) = 0, w_x(t,0) = w_x(t,L) = 0, \;\; 0 \le t. \end{cases} \tag{1}$$

The nonlinear FPHTPDE model (1) describes wave propagation in geometric and physical systems exhibiting anomalous propagation and other complex phenomena. In this expanded form of the classical hyperbolic telegraph equation, the time derivative is replaced by the Caputo fractional derivative of order α, where $0 \le \alpha < 1$.

Definition 1.1. *As stated by reference [18], the Caputo fractional derivative of α-order can be defined as follows:*

$$\frac{D^\alpha w(t,x)}{\partial t^\alpha} = D_t^\alpha w(t,x) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{1}{(t-p)^{\alpha-n+1}} \frac{\partial^n w(p,x)}{\partial p^n} dp, (n-1 < \alpha < n) \tag{2}$$

and for $\alpha = n \in N$ defined as:

$$D_t^\alpha w(t,x) = \frac{D^\alpha w(t,x)}{\partial t^\alpha} = \frac{D^n w(t,x)}{\partial t^n}.$$

2. Exact Solution and Stability Estimates for Abstract Form

This section presents a detailed study of the exact solution and stability estimates for the nonlinear fractional-order telegraph equation. The nonlinear telegraph equation is an important mathematical model with numerous applications in engineering, communications, and physics. Understanding the stability properties and exact solutions of this equation is essential for comprehending the behavior of the dynamic systems it governs. Using in-depth analysis and mathematical techniques, we aim to light up the fundamental properties underlying this fascinating set of partial differential equations and to explain the complex relationship between nonlinearity and wave propagation. This capability not only strengthens the theoretical foundations of applied mathematics but also paves the way for practical applications in a wide range of technical and scientific fields.

We consider the following fractional order nonlinear telegraph (PDE)

$$\begin{cases} w_{tt}(t) + Aw_t(t) + Aw(t) + w^2(t) + D_t^{\alpha} w(t) = f(t), \\ w(0) = \varphi,\ w'(0) = \psi. \end{cases} \tag{3}$$

We investigate a Cauchy problem for the telegraph equation

$$\begin{cases} w_{tt}(t) + Aw_t(t) + Aw(t) = G(t), 0 \le t \le T \\ w(0) = \varphi,\ w'(0) = \psi. \end{cases} \tag{4}$$

In this case, a self-adjoint positive definite operator A and $A \ge \delta I$ exist in a Hilbert space H with $\delta > 0$.

The initial-value issue for a system of first-order differential equations is identical to equation (4) and can be written in this way

$$\begin{cases} w'(t) + \frac{A}{2} w(t) + iB^{\frac{1}{2}} w(t) = z(t), 0 \le t \le T \\ z'(t) + \frac{A}{2} z(t) + iB^{\frac{1}{2}} z(t) = G(t), \end{cases} \tag{5}$$

here,

$$G(t) = F(t) - w^2(t). \tag{6}$$

By integrating the formula (5), we obtain

$$w(t) = e^{-\left(\frac{A}{2} + iB^{\frac{1}{2}}\right)t} w(0) + \int_0^t e^{-\left(\frac{A}{2} + iB^{\frac{1}{2}}\right)(t-s)} z(s)\, ds,$$

$$z(t) = e^{-\left(\frac{A}{2} - iB^{\frac{1}{2}}\right)t} z(0) + \int_0^t e^{-\left(\frac{A}{2} + iB^{\frac{1}{2}}\right)(t-s)} G(s)\, ds.$$

Applying the initial condition $z(0) = w'(0) + \left(\frac{A}{2} + iB^{\frac{1}{2}}\right) w(0)$, then we have

$$w(t) = e^{-\left(\frac{A}{2} + iB^{\frac{1}{2}}\right)t} w(0) + \int_0^t e^{-\left(\frac{A}{2} - iB^{\frac{1}{2}}\right)(t-s)} \int_0^s e^{-\left(\frac{A}{2} - iB^{\frac{1}{2}}\right)(s-p)} G(p) dp ds$$
$$+ \int_t^s e^{-\left(\frac{A}{2} + iB^{\frac{1}{2}}\right)(t-s)} . e^{-\left(\frac{A}{2} + iB^{\frac{1}{2}}\right)t} ds \left(w'^{(0)} + \left(\frac{A}{2} + iB^{\frac{1}{2}}\right) w(0) \right).$$

The above expression can be represented by switching the order of integration as:

$$(t) = \left[e^{-\left(\frac{A}{2} + iB^{\frac{1}{2}}\right)t} + \left(\frac{A}{2} + iB^{\frac{1}{2}}\right) \int_0^t e^{-\left(\frac{A}{2} + iB^{\frac{1}{2}}\right)(t-s)} e^{-\left(\frac{A}{2} - iB^{\frac{1}{2}}\right)s} ds \right] w(0)$$
$$+ \int_0^t e^{-\left(\frac{A}{2} + iB^{\frac{1}{2}}\right)(t-s)} e^{-\left(\frac{A}{2} - iB^{\frac{1}{2}}\right)s} ds w'^{(0)} + \int_0^t e^{-\frac{A}{2}(t-s)} B^{-\frac{1}{2}} \frac{e^{i(t-s)B^{\frac{1}{2}}} - e^{-i(t-s)B^{\frac{1}{2}}}}{2i} G(s) ds$$

$$= e^{-\frac{A}{2}t} \left[\frac{e^{itB^{\frac{1}{2}}} - e^{-itB^{\frac{1}{2}}}}{2} + \frac{A}{2} B^{-\frac{1}{2}} \frac{e^{itB^{\frac{1}{2}}} - e^{-itB^{\frac{1}{2}}}}{2i} \right] w(0) + e^{-\frac{A}{2}t} \left[B^{-\frac{1}{2}} \frac{e^{itB^{\frac{1}{2}}} - e^{-itB^{\frac{1}{2}}}}{2i} \right] w'(0) +$$

$$\int_0^t e^{-\frac{A}{2}(t-s)} B^{-\frac{1}{2}} \frac{e^{i(t-s)B^{\frac{1}{2}}} - e^{-i(t-s)B^{\frac{1}{2}}}}{2i} G(s) ds. \tag{7}$$

We can obtain the following from formula (7):

$$w(t) = e^{-\frac{A}{2}t}\left[\left(c(t) + s(t)\right)w(0) + s(t)w'(0)\right] + \int_0^t e^{-\frac{A}{2}(t-s)}s(t-z)G(z)dz, \tag{8}$$

here, $c(t) = \frac{e^{itB^{\frac{1}{2}}} - e^{-itB^{\frac{1}{2}}}}{2}$, $s(t) = B^{-\frac{1}{2}}\frac{e^{itB^{\frac{1}{2}}} - e^{-itB^{\frac{1}{2}}}}{2i}$, and $B = A - \frac{A^2}{4}$.

Utilizing equation (6), we have

$$w(t) = e^{-\frac{A}{2}t}\left[\left(c(t) + \frac{A}{2}s(t)\right)w(0) + s(t)w'(0)\right] + \int_0^t e^{-\frac{A}{2}(t-z)}s(t-z)F(z)dz - \int_0^t e^{-\frac{A}{2}(t-z)}s(t-z)w^2(z)dz.$$

From the last formula, we obtain

$$w(t) + \int_0^t e^{-\frac{A}{2}(t-z)}s(t-z)w^2(z)dz = e^{-\frac{A}{2}t}\left[\left(c(t) + \frac{A}{2}s(t)\right)w(0) + s(t)w'(0)\right] + \int_0^t e^{-\frac{A}{2}(t-z)}s(t-z)F(z)dz,$$

here,

$$F(z) = f(z) - {}_0^c w^{\alpha}(t). \tag{9}$$

Using the formula (7), we can get the mild solution of equation (3) as follows

$$(t) + \int_0^t e^{-\frac{A}{2}(t-z)}s(t-z)w^2(z)dz + \int_0^t e^{-\frac{A}{2}(t-z)}s(t-z){}_0^c w^{\alpha}(z)dz = e^{-\frac{A}{2}t}\left[\left(c(t) + \frac{A}{2}s(t)\right)w(0) + s(t)w'(0)\right] + \int_0^t e^{-\frac{A}{2}(t-z)}s(t-z)F(z)dz. \tag{10}$$

Now, the following theorem is presented about formula (10).

Theorem 2.1. *Suppose that $\varphi \in D(A)$, $\psi \in D(A^{\frac{1}{2}})$, and the function $f(t)$ are continuously differentiable along the interval $[0, T]$, we can say that there is a unique solution to problem (10) that satisfies the following stability inequalities.*

$$\max_{0\le t\le T}\|w(t)\|_H + \max_{0\le t\le T}\left\|A^{-\frac{1}{2}}w^2(t)\right\|_H + \left\|A^{-\frac{1}{2}}D_t^{\alpha}(t)\right\|_H \le M\left[\|\varphi\|_H + \left\|A^{-\frac{1}{2}}\psi\right\|_H + \max_{0\le t\le T}\left\|A^{-\frac{1}{2}}f(t)\right\|\right].$$

Proof. Using the formula (10), $A \ge \delta I > 0$ and the following estimates

$\|s(t)\|_{H\to H} \le 1$, $\left\|B^{\frac{1}{2}}s(t)\right\|_{H\to H} \le 1$, $\left\|e^{-\frac{A}{2}(t-z)}\right\|_{H\to H} \le 1$, $\left\|e^{-\frac{A}{2}}\varphi\right\|_{H\to H} \le \frac{1}{\delta}\|\psi\|_H$, and $\left\|\frac{A}{2}B^{-\frac{1}{2}}\right\|_{H\to H} \le M(\delta)$,

We can write the following inequalities

$$\|w(t)\|_H + \int_0^t \left\|e^{-\frac{A}{2}(t-z)}\right\|_{H\to H}\left\|B^{\frac{1}{2}}s(t-z)\right\|_{H\to H}\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}w^2\right\|_{H\to H}dz$$

$$+\int_0^t \left\|e^{-\frac{A}{2}(t-z)}\right\|_{H\to H}\left\|B^{\frac{1}{2}}s(t-z)\right\|_{H\to H}\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}D_t^{\alpha}(t)\right\|_{H\to H}dz$$

$$\leq \|s(t)\|_{H\to H}\left\|e^{-\frac{A}{2}}\varphi\right\|_{H\to H}+\left\|B^{\frac{1}{2}}s(t)\right\|_{H\to H}\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\times\left|\frac{A}{2e^{\frac{A}{2}}}\right|\left\|e^{-\frac{A}{2}}\varphi\right\|_{H\to H}$$
$$+\left\|B^{\frac{1}{2}}s(t)\right\|_{H\to H}\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}e^{-\frac{A}{2}t}\left\|e^{-\frac{A}{2}}\psi\right\|_{H\to H}$$
$$+\int_0^t \left\|e^{-\frac{A}{2}(t-z)}\right\|_{H\to H}\left\|B^{\frac{1}{2}}s(t-z)\right\|_{H\to H}\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}$$
$$\times\left\|A^{-\frac{1}{2}}f(s)\right\|_{H\to H}ds, \text{for any } t\in[0,T].$$

Then, we obtain

$$\max_{0\leq t\leq T}\|w(t)\|_H+\max_{0\leq t\leq T}\left\|A^{-\frac{1}{2}}w^2(t)\right\|_H+\max_{0\leq t\leq T}\left\|A^{-\frac{1}{2}}D_t^{\alpha}w(t)\right\|_H$$
$$\leq M(\delta)\left[\|\varphi\|_H+\left\|A^{-\frac{1}{2}}\psi\right\|_H+\max_{0\leq t\leq T}\left\|A^{-\frac{1}{2}}f(t)\right\|\right].$$

Following proof of Theorem 2.1, we will now examine its applications in more detail. Let us begin by considering the boundary value problem of the fractional-order telegraph equation. This topic is of great importance due to its numerous applications, which include heat transfer in porous media and the dynamic simulation of viscoelastic materials.

The proposed theorem will be utilized to create a series of approximations that will lead to the exact solution in order to solve this problem. Along with various numerical examples, we will also talk about the solution's uniqueness.

In addition, we explore some applications of Theorem 2.1, including solving fractional-order integral equations and initial value problems for fractional-order differential equations. These schemes are added to boundary value problems for the fractional-order telegraph equations.

Finally, by solving the initial value problem (3) in Hilbert space $H=L_2[0,L]$ using the given operator A^x, we can obtain the singular smooth solution to problem (1). This is because problem (1) has a unique smooth solution $w(t,x)$, for $x\in(0,L)$, and $\delta>0$. This is because both $\varphi(x)$ and $\psi(x)$ are smooth. Using this method, we can simplify the problem to the initial value of (3) in Hilbert space using the positive self-adjoint operator A^x, denoted by

$$A^x w(x)=w_{xx}(x)+\delta w(x), \tag{11}$$

We can write the following expression $(A^x w_{xx})_t=-w_{xxt}=-w_{txx}$, for the continuous $w(t,x)$, with the domain

$$D(A^x)=\{w(x): w, w_x, w_{xx}\in L_2[0,L], w(0)=w(L), w'(0)=w'(L)\},$$

here, $f(t)=f(t,x)$, $w(t)=w(t,x)$, and $D_t^{\alpha}w(t)=D_t^{\alpha}w(t,x)$ are known and unknown functions defined on $[0,L]$ and taking values in the $H=L_2[0,L]$ with the norm $\|f\|_{L_2[0,L]}=\{\int|f(x)|^2dx\}^{\frac{1}{2}}$. ■

3. Projected Scheme and Stability Result

This section explores the development and analysis of finite difference schemes designed for nonlinear FPHTE. It aims to enhance our understanding of computational methods used to handle complex mathematical models and to evaluate the validity of these numerical methods in calculating FPHTE by analysing their stability characteristics. Finite difference methods should be explored in this context to improve our ability to model and analyse real-world processes governed by fractional differential equations.

We discretize both the spatial and temporal domains to apply the explicit finite difference approach to model (1). Let w_n^k be a numerical approximation of (x_n, t_k), with x_n representing the spatial grid point and t_k representing the temporal level. The approximation of the fractional derivative ${}_0^C D_t^\alpha w(t,x)$ using finite difference is accomplished by applying the Caputo fractional derivative.

To apply the explicit finite difference method to model (1), we separate the spatial and temporal domains. Suppose w_n^k is a numerical approximation of (x_n, t_k), where t_k is the time plane and x_n is a point on the spatial grid. The Caputo fractional derivative is used to approximate the fractional derivative ${}_0^C D_t^\alpha w(t,x)$ using finite differences.

For the difference scheme method, let's start by assuming that $h = \frac{L}{M}$ for x-axis and $\tau = \frac{T}{N}$ for t-axis as a grid throughout the w-shaped domain, $w^h \times w^\tau$. Then,

$x_n = nh, n = 1,2,\cdots,M, \quad t_k = k\tau, k = 1,2,\cdots,N.$

At the positions $(t_k, x_n) \in w^h \times w^\tau$, we may express the issue (1) as

$$w_{tt}(t_k,x_n) + {}_0^C D_t^\alpha w(t_k,x_n) + w^2(t_k,x_n) = \lambda u_{txx}(t_k,x_n) + u_{xx}(t_k,x_n) + f(t_k,x_n). \tag{12}$$

What follows is the procedure for developing the basic first-order difference technique is described for each individual component in model (1). The formula for calculating the finite differences of Caputo fractional derivative given in [13] is as follows:

$$ {}_0^C D_t^\alpha w(t,x) = \mathrm{S}_{\alpha,\tau} \sum_{j=0}^k w_j^{(\alpha)} \left(w_n^{k-j+1} - w_n^{k-j}\right), \tag{13}$$

here $w_j^{(\alpha)} = (j+1)^{1-\alpha} - j^{1-\alpha}$ and $\mathrm{S}_{\alpha,\tau} = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}$.

This is the difference scheme formula for model (1) that we can derive from using formula (13) and generating the first-order difference technique for the remaining elements of (12):

$$\begin{cases} \frac{w_n^{k+1} - 2w_n^k + w_n^{k-1}}{\tau^2} + \mathrm{S}_{\alpha,\tau}\left(w_n^{k+1} - w_n^k\right) + \mathrm{S}_{\alpha,\tau} \sum_{j=1}^k w_j^{(\alpha)} \left(w_n^{k-j+1} - w_n^{k-j}\right) + {w_n}^2 \\ \quad = \frac{\lambda}{\tau}\left[\frac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{h^2} - \frac{w_{n+1}^{k-1} - 2w_n^{k-1} + w_{n-1}^{k-1}}{h^2}\right] + \frac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{h^2} + f_n^k, \\ \qquad 1 \le k \le \mathrm{N}-1, 1 \le n \le \mathrm{M}-1, \\ w_n^0 = \varphi(x_n), \ \frac{w_n^1 - w_n^0}{\tau} = \psi(x_n), \ 0 \le n \le M, \\ w_0^k = w_M^k = 0\,, \ 0 \le k \le \mathrm{N}. \end{cases} \tag{14}$$

Theorem 3.1. *If the expression* $1 - 2\left(\frac{\lambda}{h^2\tau} - \frac{1}{\tau^2} + \frac{1}{h^2}\right) - \frac{\lambda w_1}{\Gamma(2-\alpha)\tau^\alpha} > 0$.. *The equation (14) exhibits stability.*

Proof. The theorem was proved, as stated in [13], using the same procedure as in [19].

4. Numerical Investigations

Within this section, we employ the finite difference scheme (14) to compute numerical solutions for model (1) and compare them to analytical solutions to estimate errors and accuracy. We derive analytical solutions through integration and algebraic manipulation and then implement the finite difference scheme (14) to obtain numerical solutions. Then, we estimate relative errors between the numerical and analytical solutions and generate formal error bounds of the numerics to evaluate accuracy. We apply a modified Gauss elimination method to solve the linear systems for better stability and efficiency. Overall, the procedure is to derive analytical solutions, implement numerical simulations, quantify relative errors, derive error bounds, and assess validity to verify the fidelity of our numerical approach precisely. We have employed the subsequent methodology to compute the error predicts:

$$\varepsilon = \max_{\substack{k=0,1,\dots,N \\ n=0,1,\dots,M}} |w(t_k, x_n) - w(t,x)|.$$

Example 4.1. We consider the following nonlinear FPHTPDEs:

$$\begin{cases} w_{tt}(t,x) + {}_0^C D_t^{\alpha} w(t,x) + w^2(t,x) = \lambda w_{txx}(t,x) + w_{xx}(t,x) + f(t,x) \\ f(t,x) = \left(2 + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + 2(t^2 - t + 1)^2 \sin x + \lambda(2t-1) + t^2 - t + 1\right) \sin x, \\ \qquad 0 < x < \pi, t > 0, \lambda > 0, 0 < \alpha \leq 1, \\ u(0,x) = sinx, u_t(0,x) = sinx, 0 \leq x \leq \pi, \\ u(t,0) = u(t,\pi) = 0, \quad t \geq 0. \end{cases} \tag{15}$$

Table 1. Numerical results for example 1, using the explicit finite difference technique for different value of λ, α, M and N.

M, N	Max. analytical	Max. approximation	Max error	Relative error
$5, 25$	0.9511	0.9584	73×10^{-4}	63×10^{-4}
$10, 100$	1.0000	1.0020	2×10^{-3}	15×10^{-4}
$15, 225$	0.9945	0.9954	9×10^{-4}	7×10^{-4}
$20, 400$	1.0000	1.0004	4×10^{-4}	4×10^{-4}
$25, 625$	0.9980	0.9983	3×10^{-4}	3×10^{-4}
$30, 900$	1.0000	1.0001	2×10^{-4}	2×10^{-4}
$35, 1225$	0.9990	0.9993	3×10^{-4}	1×10^{-4}
$40, 1600$	0.9990	0.9993	3×10^{-4}	1×10^{-4}

Table 1 comprehensive numerical error analysis establishes trust and demonstrates alignment with analytical solutions more robustly than solution graphs, providing critical insights graphs alone cannot. The table provided important insights into accuracy and reliability that figures cannot.

The error analysis column includes extensive numerical simulations that replicate approximate responses under various conditions; these simulations form the basis for measuring accuracy. The numerical results demonstrate how approximation errors depend on grid spacing and time steps through simulations across different parameters and grid reliability. Rather than focusing solely on initial solutions, the simulations specifically aim to enable convergence testing and detect degradation. By emphasizing the need for caution in assessing numerical reliability, optimizing the simulations in this way enhances dependence and accuracy in quantifying the error. Furthermore, compiling the simulation data into tables facilitates the reproducibility of results, allowing other researchers to independently verify accuracy. The solutions are presented graphically as follows:

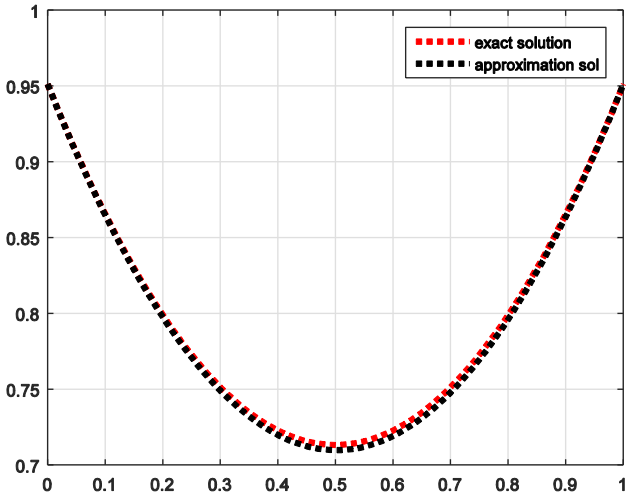


Figure 1. 2D approximate and exact solutions of equation (15), for $M = 10, N = 100, \lambda = 5,\ \alpha = 0.95$.

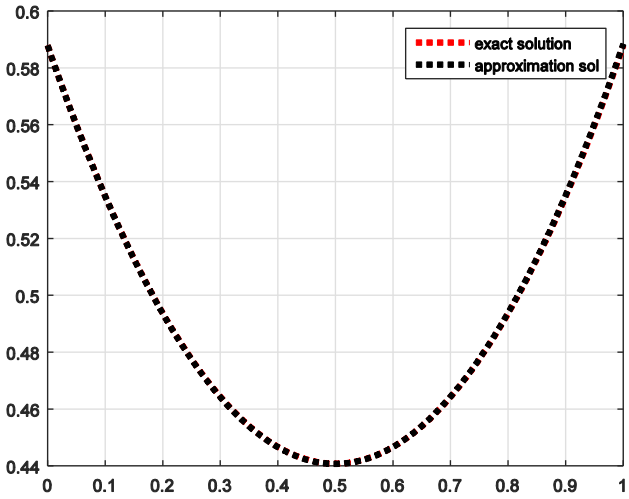


Figure 2. 2D approximate and exact solutions of equation (15), for $M = 20, N = 400, \lambda = 5,\ \alpha = 0.99$.

Figures 1–3 compare the approximate and exact solutions of equation (15) under increasing grid refinement: Figure 1 uses $M = 10, N = 100, \lambda = 5, \alpha = 0.95$; Figure 2 uses $M = 20, N = 400, \lambda = 5, \alpha = 0.99$; and Figure 3 uses $M = 35$, $N = 1225$, $\lambda = 5$, $\alpha = 0.99$. As M and N increase, the approximate solution converges more closely to the exact one, confirming the accuracy and reliability of the proposed scheme.

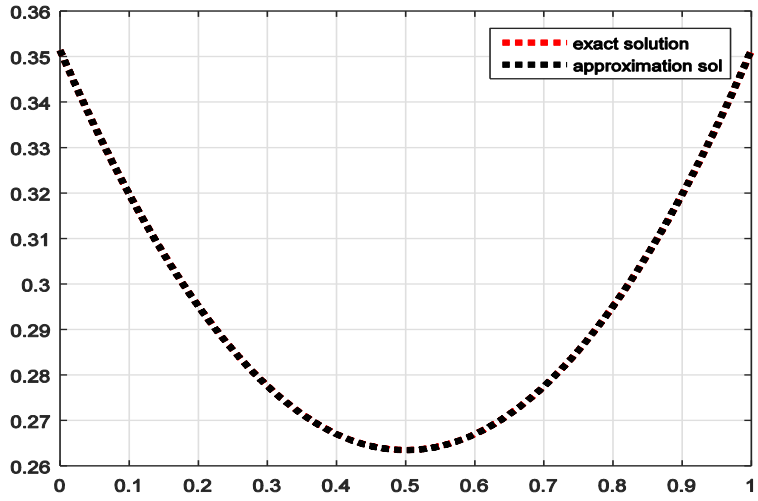


Figure 3. 2D approximate and exact solutions of equation (15), for $M = 35, N = 1225, \lambda = 5, \alpha = 0.99$.

5. Conclusion

In summary, a new theoretical framework has been developed for solving nonlinear FPHTE equations. We have devised a detailed method for efficiently solving a variety of nonlinear FPHTE equations by deriving a formula independent of specific nonlinear properties. Stability analysis has shown that the proposed method produces reliable and consistent results. This innovative method represents a significant advance in the efficient modeling of complex nonlinear dynamics described by nonlinear FPHTE equations, which are relevant to numerous application areas. Furthermore, this paper presents and investigates a method that explicitly utilizes finite differences to successfully solve nonlinear FPHTE equations. The presented method has demonstrated its reliability and effectiveness through extensive numerical experiments and its implementation using MATLAB software.

In addition to demonstrating the stability and convergence of the method across a wide range of fractional orders, our results highlight the importance of selecting appropriate time steps, spatial grid sizes, and fractional orders to achieve stability and numerical accuracy. This paper contributes to our understanding of how complex wave dynamics with memory effects can be represented using finite-difference techniques in a range of geometric and physical systems.

Therefore, our results open avenues for further research to explore and extend the proposed method for solving more complex nonlinear fractional order equations, and to enhance our understanding of wave propagation processes in various scientific and professional fields.

Declarations

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validation, writing-original draft preparation. **Ali Akgül:** Investigation, writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Data availability: The data and codes that support the findings of this study are available from the corresponding author upon reasonable request.

Use of AI tools: During the preparation of this work, the authors did not use any AI tools or AI-assisted technologies.

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